P. 01

Solutions for HWk week 8

Problem 1 (problem 22, p264)

$y' = \sqrt{1-y^2}, \quad y(0) = 0$

(a) $y = \sin x, \quad y' = \cos x, \quad \sqrt{1-y^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = \cos x$

if $y = \sin x, \quad y' = \sqrt{1-y^2}$ ※

(b) $y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \sqrt{1-y^2} = \sqrt{1-(\sum_{n=0}^{\infty} a_n x^n)^2}$

$$y^2 = \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$
$$= a_0^2 + 2a_0 a_1 x + (2a_0 a_2 + a_1^2)x^2 + (2a_1 a_2 + 2a_0 a_3)x^3 + \cdots$$

$\sqrt{1-y^2} \sim 1 - \frac{1}{2}y^2$ when $|y| \ll 1$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \sim 1 - \frac{1}{2}\left(\sum_{n=0}^{\infty} a_n x^n\right)^2$$
$$= 1 - \frac{1}{2}\left(a_0^2 + 2a_0 a_1 x + (2a_0 a_2 + a_1^2)x^2 + 2(a_1 a_2 + a_0 a_3)x^3 + \cdots\right)$$

$x^0: \quad a_1 = 1 - \frac{1}{2}a_0^2$

$x^1: \quad 2a_2 = -a_0 a_1$

$x^2: \quad 3a_3 = -\frac{1}{2}(2a_0 a_2 + a_1^2) = -\left(a_0 a_2 + \frac{a_1^2}{2}\right)$

$x^3: \quad 4a_4 = -\frac{2}{2}\cdot(a_1 a_2 + a_0 a_3)$

$y(0) = 0 \Rightarrow a_0 = 0$

$a_1 = 1$

$a_2 = 0$

$a_3 = -\frac{1}{6}$

$a_4 = 0$

$$\boxed{y \sim x - \frac{1}{6}x^3}$$

Problem 2 (problem 35, p 280)

$x^2 y'' + \alpha x y' + \frac{5}{2}y = 0$

$y = x^r, \quad r(r-1) + \alpha r + \frac{5}{2} = 0$

$r^2 + (\alpha-1)r + \frac{5}{2} = 0, \quad r = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\times\frac{25}{4}}}{2} = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 25}}{2}$

$r_1 = \frac{-(\alpha-1) + \sqrt{(\alpha-1)^2 - 25}}{2}$

$r_2 = \frac{-(\alpha-1) - \sqrt{(\alpha-1)^2 - 25}}{2}$

if $(\alpha-1)^2 - 25 \neq 0$

$y_1 = x^{r_1}$ & $y_2 = x^{r_2}$

want α such that $y_1 \to 0$ as $x \to 0$
$y_2 \to 0$

this implies that $-\frac{(\alpha-1)}{2} > 0, \quad \alpha - 1 < 0, \quad \boxed{\alpha < 1}$

Problem 3 (problem 36 p 280)

$x^2y'' + \beta y = 0$

$y = x^r$, $r(r-1) + \beta = 0$, $r^2 - r + \beta = 0$

$r = \frac{1 \pm \sqrt{1-4\beta}}{2}$, want $r > 0$ such that $y \to 0$ as $x \to 0$

$1 > 1 - 4\beta$, $\boxed{\beta > 0}$

Problem 4 (Problem 37 P 281)

$x^2y'' - 2y = 0$, $y(1) = 1$, $y'(1) = \gamma$

$y = x^r$, $r(r-1) - 2 = 0$, $r^2 - r - 2 = 0$ $(r-2)(r+1) = 0$

$r = -1, 2$, $y = C_1 x^{-1} + C_2 x^2$ $y(1) = C_1 + C_2 = 1$

$y' = -C_1 x^{-2} + 2C_2 x$ $y'(1) = -C_1 + 2C_2 = \gamma$

$3C_2 = 1 + \gamma$, $C_2 = \frac{1+\gamma}{3}$

$C_1 = 1 - C_2 = 1 - \frac{1+\gamma}{3} = \frac{2-\gamma}{3}$

$y(x) = \frac{2-\gamma}{3} x^{-1} + \frac{1+\gamma}{3} x^2$ bounded as $x \to 0$ means $2 - \gamma = 0$, $\boxed{\gamma = 2}$