Math 222, Fall 2016.

Present your work in an organized fashion. Make sure that your work is algebraically correct and logically sound. Show all your work. No calculator, notes, or books.

Quiz 09/19/2016 M222-001

1. Find the solution to the initial value problem (IVP):

$$y' = y^2 - 1, \quad y(0) = -2.$$

2. A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at a rate of 3 L/min. When is the tank empty? Find an expression for the amount of salt in the tank at any time before the tank is empty.

1. $y' = y^2 - 1$, $\frac{dy}{y^2-1} = dt$, $\int \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = \int dt$, $\ln\left|\frac{y-1}{y+1}\right| = 2t + C$

$\left|\frac{y-1}{y+1}\right| = Ae^{2t}$, $y(0) = -2$, this means that $y'(0) > 0$. This implies that y will evolve from -2 toward -1. Therefore $\left|\frac{y-1}{y+1}\right| = \frac{y-1}{y+1}$

$\frac{y-1}{y+1} = Ae^{2t}$, $y - 1 = Ae^{2t}y + Ae^{2t}$, $(1 - Ae^{2t})y = 1 + Ae^{2t}$,

$y = \frac{1+Ae^{2t}}{1-Ae^{2t}}$, $y(0) = -2 = \frac{1+A}{1-A}$, $-2 + 2A = 1 + A$, $\boxed{A = 3}$

$$\boxed{y = \frac{1+3e^{2t}}{1-3e^{2t}}}$$

2. $\frac{dV}{dt} = 2 - 3 \text{ L/min} = -1 \text{ L/min}$, $V = -t + V_0$, $V_0 = 120 \text{ L}$,

$Q' = 2\cdot\gamma - 3\cdot\frac{Q}{120-t}$, $Q' + \frac{3}{120-t}Q = 2\gamma$, $\mu = e^{\int \frac{3}{120-t}dt} = e^{-3\ln(120-t)}$,

$\mu = \frac{1}{(120-t)^3}$, $Q = \frac{\int 2\gamma \cdot \frac{1}{(120-t)^3}dt + C}{\frac{1}{(120-t)^3}} = \frac{\gamma(120-t)^2 + C}{\frac{1}{(120-t)^3}} = \gamma(120-t) + C(120-t)^3$

$Q(0) = 0 = \gamma \cdot 120 + C \cdot 120^3$, $C = -\frac{120\gamma}{120^3} = -\frac{\gamma}{14400}$

$$\boxed{Q(t) = \gamma(120-t) - \frac{\gamma}{14400}(120-t)^3}$$

Tank is empty when $V = 0 = 120 - t$, $\boxed{t = 120 \text{ minutes}}$