Solution to practice problems for final exam

Problem 1 (problem 22 page 367)

$$\frac{dV_1}{dt} = 1.5 + 1.5 - 3.0 = 0, \quad V_1 = \text{constant} = 30 \text{ gal.}$$

$$\frac{dV_2}{dt} = 1 + 3 - 1.5 - 2.5 = 0, \quad V_2 = \text{constant} = 20 \text{ gal}$$

$$\frac{dQ_1}{dt} = 1.5 \times 1 + 1.5 \times \frac{Q_2}{20} - 3 \times \frac{Q_1}{30}$$

$$\frac{dQ_2}{dt} = 1 \times 3 + 3 \times \frac{Q_1}{30} - 4 \times \frac{Q_2}{20}$$

$$\begin{cases} \dfrac{dQ_1}{dt} = 1.5 - \dfrac{Q_1}{10} + \dfrac{3}{40} Q_2 \\[2ex] \dfrac{dQ_2}{dt} = 3 + \dfrac{Q_1}{10} - \dfrac{1}{5} Q_2 \end{cases}$$

at equilibrium $\quad Q_1' = 0 = 1.5 - \frac{Q_1}{10} + \frac{3}{40} Q_2 \quad$ — ①

$$Q_2' = 0 = 3 + \frac{Q_1}{10} - \frac{1}{5} Q_2 \quad \text{— ②}$$

① + ②, $\quad 0 = 4.5 + \frac{3-8}{40} Q_2, \quad \frac{1}{8} Q_2 = 4.5, \quad Q_2 = 36$

$$1.5 - \frac{Q_1}{10} + \frac{27}{10} = 0, \quad Q_1 = 42$$

$$x_1 = Q_1 - Q_1^E = Q_1 - 42$$

$$x_2 = Q_2 - Q_2^E = Q_2 - 36$$

$$\frac{dQ_1}{dt} = \frac{dx_1}{dt} = 1.5 - \frac{x_1 + 42}{10} + \frac{3}{40}(x_2 + 36) = -\frac{1}{10} x_1 + \frac{3}{40} x_2$$

$$\frac{dQ_2}{dt} = \frac{dx_2}{dt} = 3 + \frac{x_1 + 42}{10} - \frac{1}{5}(x_2 + 36) = \frac{1}{10} x_1 - \frac{1}{5} x_2$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix}$$

$$\det(A-\lambda I)=0, \qquad \det\begin{pmatrix} -\frac{1}{10}-\lambda & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5}-\lambda \end{pmatrix}=0$$

$$\left(-\frac{1}{10}-\lambda\right)\left(-\frac{1}{5}-\lambda\right)-\frac{3}{400}=0$$

$$400\cdot\left(\lambda+\frac{1}{10}\right)\left(\lambda+\frac{1}{5}\right)-3=0$$

$$(20\lambda+1)(4\lambda+1)=0 \qquad \lambda=-\frac{1}{20},\ -\frac{1}{4}$$

$\lambda=-\frac{1}{20}$

$$\begin{pmatrix} -\frac{1}{20} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5}+\frac{1}{20} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -\frac{1}{20}x_1+\frac{3}{40}x_2=0, \quad -2x_1+3x_2=0$$

$$2x_1=3x_2, \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\lambda=-\frac{1}{4}$

$$\begin{pmatrix} \frac{3}{20} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5}+\frac{1}{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 2x_1+x_2=0, \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=c_1\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{-\frac{t}{20}}+c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-\frac{t}{4}}$$

at $t=0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 25-42 \\ 15-36 \end{pmatrix}=\begin{pmatrix} -17 \\ -11 \end{pmatrix}=c_1\begin{pmatrix} 3 \\ 2 \end{pmatrix}+c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$3c_1+c_2=-17 \qquad c_1=\frac{-45}{8}$$

$$2c_1-2c_2=-11 \qquad c_2=-\frac{1}{8}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=-\frac{45}{8}\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{-\frac{t}{20}}-\frac{1}{8}\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-\frac{t}{4}}=\begin{pmatrix} Q_1-42 \\ Q_2-36 \end{pmatrix}$$

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}=\begin{pmatrix} 42 \\ 36 \end{pmatrix}-\frac{45}{8}\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{-\frac{t}{20}}-\frac{1}{8}\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-\frac{t}{4}}$$

Problem 2 Find the solution of the given IVP.

$y'' + 4y = g(t), \quad y(0) = 0, \quad y'(0) = 0$

where $g(t) = \begin{cases} 1 & t < \pi \\ \sin t & t \geq \pi \end{cases}$

$$g(t) = 1 + (\sin t - 1) u_\pi$$

$$\mathcal{L}[y'' + 4y] = \mathcal{L}[g] = \mathcal{L}[1 + (\sin t - 1) u_\pi]$$

$$\sin t = \sin(t - \pi + \pi) = \sin(t-\pi)\cos\pi + \cos(t-\pi)\sin\pi$$
$$= -\sin(t-\pi)$$

$$(s^2+4)Y = \frac{1}{s} + \mathcal{L}[\sin t \cdot u_\pi] - \frac{e^{-\pi s}}{s}$$

$$= \frac{1}{s} - \mathcal{L}[\sin(t-\pi) u_\pi] - \frac{e^{-\pi s}}{s}$$

$$(s^2+4)Y = \frac{1}{s} - \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s}$$

$$Y = \frac{1}{s(s^2+4)} - \frac{e^{-\pi s}}{(s^2+1)(s^2+4)} - \frac{e^{-\pi s}}{s(s^2+4)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{Bs+C}{s^2+4}\right]$$

$A(s^2+4) + Bs^2 + Cs = 1, \quad 4A = 1, \quad A = \frac{1}{4},$

$A + B = 0, \quad B = -A = -\frac{1}{4}$

$C = 0$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+2^2}\right] = \frac{1}{4} - \frac{1}{4}\cos 2t$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] = \frac{1}{3}\left(\sin t - \frac{1}{2}\sin 2t\right)$$

$$\therefore\ y(t) = \frac{1}{4} - \frac{1}{4}\cos 2t - u_\pi \cdot \left(\frac{1}{4} - \frac{1}{4}\cos 2(t-\pi)\right)$$

$$- \frac{1}{3} u_\pi \left(\sin(t-\pi) - \frac{1}{2}\sin 2(t-\pi)\right)$$

Problem 3 Find the solution of the IVP $ty' = \frac{1}{y+1}$, $y(1) = 0$.
For what t-interval is the solution defined?

$(y+1)y' = \frac{1}{t}$, $\quad \frac{1}{2}(y+1)^2 = \ln t + C$

$(y+1)^2 = 2\ln t + 2C$, $\quad t=1,\ y(1)=0,\ 1^2 = 2\cdot\ln 1 + 2\cdot C$

$C = \frac{1}{2}$

$(y+1)^2 = 2\ln t + 1$

$y+1 = \pm\sqrt{2\ln t + 1}$ $\quad y(1) = 0$, need to choose "+"

$y+1 = \sqrt{2\ln t+1}$, $\quad \boxed{y = \sqrt{2\ln t+1} - 1}$

$2\ln t + 1 > 0$, $\quad \ln t > -\frac{1}{2}$, $\boxed{\infty > t > e^{-\frac{1}{2}}}$

Problem 4 (i) Evaluate the following integral $\int_0^\infty e^{-(s+3)t} \cdot \sin 2t \, dt$

let $\bar{s} = s+3$, $\int_0^\infty e^{-(s+3)t} \sin 2t \, dt = \int_0^\infty e^{-\bar{s}t} \sin 2t \, dt$

$$\Rightarrow \mathcal{L}[\sin 2t] = \int_0^\infty e^{-st} \sin 2t \, dt = \frac{2}{s^2+2^2}$$

$$\int_0^\infty e^{-(s+3)t} \sin 2t \, dt = \frac{2}{(s+3)^2+2^2}$$

(ii) Find the inverse Laplace transform of $F(s) = e^{-3s} \cdot \frac{16}{(s+2)(s^2-4)}$

$$\mathcal{L}^{-1}\left[\frac{16}{(s+2)(s^2-4)}\right] = \mathcal{L}^{-1}\left[\frac{16}{(s+2)^2(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{As+B}{(s+2)^2} + \frac{C}{s-2}\right]$$

$$(As+B)(s-2) + C(s+2)^2 = 16$$

$s=2$, $\quad C \cdot 4^2 = 16$, $C = 1$

$s=-2$ $\quad (-2A+B)(-4) = 16$, $\quad -2A+B = -4$

$$(As+B)(s-2) + C(s+2)^2 = 16$$

s^2 terms: $\quad A + C = 0$, $\quad A = -C = -1$, $\quad B = -4+2A = -6$

$$\mathcal{L}^{-1}\left[\frac{16}{(s+2)^2(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{-s-6}{(s+2)^2} + \frac{1}{s-2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{-(s+2)-4}{(s+2)^2} + \frac{1}{s-2}\right] = \mathcal{L}^{-1}\left[-\frac{1}{s+2} - \frac{4}{(s+2)^2} + \frac{1}{s-2}\right]$$

$$= -e^{-2t} - t \cdot e^{-2t} + e^{2t}$$

$$\therefore \mathcal{L}^{-1}[F] = u_3\left(-e^{-2(t-3)} - (t-3)\, e^{-2(t-3)} \cdot 4 + e^{2(t-3)}\right)$$

Problem 5 Find particular solutions of the following ODE's

(a) $y'' - y' + 3y = \sin t$

homogeneous solution: $r^2 - r + 3 = 0$, $r = \frac{1 \pm \sqrt{1-12}}{2} = \frac{1 \pm 11i}{2}$

particular solution

$Y = A\sin t + B\cos t$,

$Y' = A\cos t - B\sin t$

$Y'' = -A\sin t - B\cos t$

$(-A\sin t - B\cos t) - (A\cos t - B\sin t) + 3(A\sin t + B\cos t) = \sin t$

$\sin t$: $-A + B + 3A = 1$ $\qquad 2A + B = 1$, $5B = 1$, $B = 1/5$

$\cos t$: $-B - A + 3B = 0$ $\qquad -A + 2B = 0$, $A = 2B$

$$\boxed{Y = \frac{2}{5}\sin t + \frac{1}{5}\cos t}$$

(b) $y'' + 2y' - 3y = e^t$

homogeneous solution: $r^2 + 2r - 3 = 0$, $(r+3)(r-1) = 0$, $r = -3, 1$

$Y = Ate^t$, $Y' = Ae^t + Ate^t$, $Y'' = Ae^t + Ae^t + Ate^t$

$Ate^t + 2Ae^t + 2(Ae^t + Ate^t) - 3Ate^t = e^t$

$4Ae^t = e^t$, $A = \frac{1}{4}$

$$\boxed{Y = \frac{t}{4}e^t}$$