P.01

Solution to practice problem for final exam

Problem 1 Find the eigenvalues λ and eigenfunctions of the boundary value problem for a positive constant L

$$y''+\lambda y=0, \quad y(0)=y(L), \quad y'(0)=y'(L)$$

$\lambda=0$, $\quad y''=0, \quad y=ax+b, \quad y'=a$

$y(0)=y(L) \Rightarrow a\cdot 0+b=a\cdot L+b, \quad a=0.$

$y'(0)=y'(L) \Rightarrow \quad 0=0 \checkmark$

$\Rightarrow \; y=1$ is an eigenfunction for $\lambda=0$

$\lambda>0$, $\quad y''+\lambda y=0, \quad y=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$

$y'=\sqrt{\lambda}\,(-C_1\sin\sqrt{\lambda}x+C_2\cos\sqrt{\lambda}x)$

$y(0)=C_1=y(L)=C_1\cos\sqrt{\lambda}L+C_2\sin\sqrt{\lambda}L$

$y'(0)=\sqrt{\lambda}\cdot C_2=y'(L)=\sqrt{\lambda}\,(-C_1\sin\sqrt{\lambda}L+C_2\cos\sqrt{\lambda}L)$

$(1-\cos\sqrt{\lambda}L)C_1-\sin\sqrt{\lambda}L\,C_2=0$

$\sin\sqrt{\lambda}L\,C_1+(1-\cos\sqrt{\lambda}L)C_2=0$

$$\begin{pmatrix}1-\cos\sqrt{\lambda}L & -\sin\sqrt{\lambda}L\\ \sin\sqrt{\lambda}L & 1-\cos\sqrt{\lambda}L\end{pmatrix}\begin{pmatrix}C_1\\C_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$(1-\cos\sqrt{\lambda}L)^2+\sin^2\sqrt{\lambda}L=0$

$2=2\cos\sqrt{\lambda}L, \quad \cos\sqrt{\lambda}L=1, \quad \sqrt{\lambda}L=2n\pi,$

$\sqrt{\lambda}=\frac{2n\pi}{L}$

$\lambda=\left(\frac{2n\pi}{L}\right)^2$

Corresponding eigenfunctions $\cos\frac{2n\pi x}{L}$

$\sin\frac{2n\pi x}{L}$

$\lambda<0 \quad y''-|\lambda|y=0. \quad y=C_1e^{\sqrt{|\lambda|}x}+C_2e^{-\sqrt{|\lambda|}x}$

$$y' = \sqrt{|\lambda|}\cdot\left(c_1 e^{\sqrt{|\lambda|}x} - c_2 e^{-\sqrt{|\lambda|}x}\right)$$

$y(0) = y(L) \Rightarrow \quad c_1 + c_2 = c_1 e^{\sqrt{|\lambda|}L} + c_2 e^{-\sqrt{|\lambda|}L}$

$y'(0) = y'(L) \Rightarrow \quad c_1 - c_2 = c_1 e^{\sqrt{|\lambda|}L} - c_2 e^{-\sqrt{|\lambda|}L}$

$$\left(1-e^{\sqrt{|\lambda|}L}\right)c_1 + \left(1-e^{-\sqrt{|\lambda|}L}\right)c_2 = 0$$
$$\left(1-e^{\sqrt{|\lambda|}L}\right)c_1 - \left(1-e^{-\sqrt{|\lambda|}L}\right)c_2 = 0$$

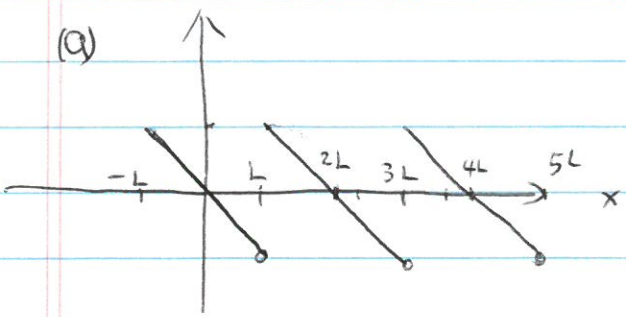
$c_1 = c_2 = 0 \Rightarrow \lambda < 0$ is not an eigenvalue

Problem 2: $f(x) = -x, \quad -L \le x < L, \quad f(x+2L) = f(x)$

(a) Sketch the graph of $f(x)$ for three periods.

(b) Find Fourier series for $f(x)$.

(a)



(b) $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-L}^{L} -x\,dx = \frac{-1}{2L} L^2 \Big|_{-L}^{L} = 0$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{L} -x\cos\frac{n\pi x}{L}\,dx = 0$ because

$x\cdot\cos\frac{n\pi x}{L}$ is odd for $x \in [-L, L]$

$$b_n = \frac{1}{L}\int_{-L}^{L} -x\cdot\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\left(-x\cdot\left(-\frac{1}{\frac{n\pi}{L}}\cos\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \int_{-L}^{L} -1\cdot\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)dx\right)$$

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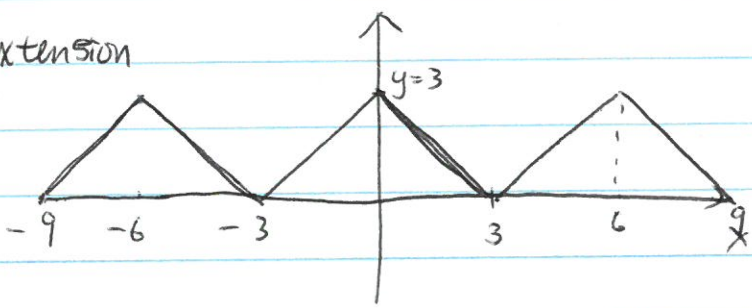
$$b_n = \frac{1}{L}\left(\frac{x \cdot L}{n\pi} \cos\frac{n\pi x}{L}\Big|_{-L}^{L} - \frac{L}{n\pi} \cdot \frac{L}{n\pi} \sin\frac{n\pi x}{L}\Big|_{-L}^{L} \right)$$

$$b_n = \frac{L^2}{L n\pi}\left(\cos n\pi + \cos n\pi\right) = \frac{2L}{n\pi}(-1)^n$$

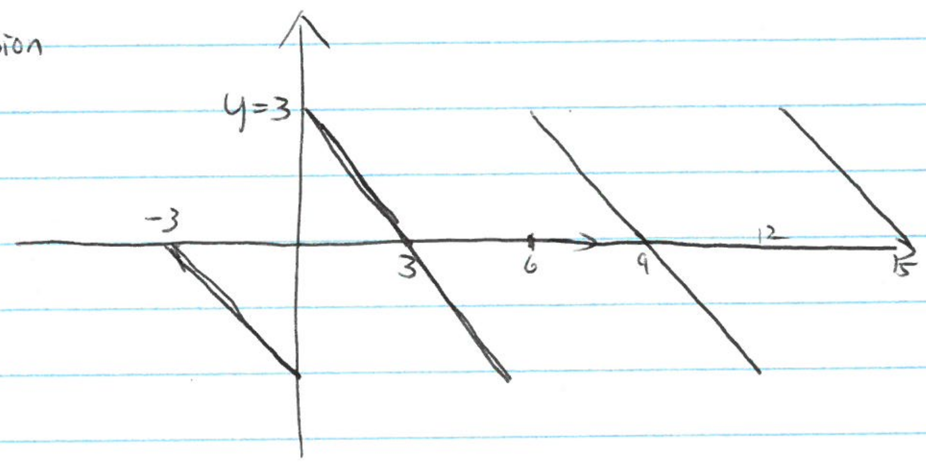
Problem 3: (problem 27 P621)

$f(x) = 3 - x, \quad 0 < x < 3$

even extension



odd extension



For even extension $b_n = 0$,

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{2}{3}\int_0^3 3 - x\,dx = \frac{2}{3}\left(3 - \frac{x^2}{2}\right)\Big|_0^3 = -1$$

$$a_n = \frac{2}{3}\int_0^3 (3-x)\cos\frac{n\pi x}{3}\,dx = 2\int_0^3 \cos\frac{n\pi x}{3}\,dx - \frac{2}{3}\int_0^3 x\cos\frac{n\pi x}{3}\,dx$$

$$= 2 \cdot \frac{1}{\frac{n\pi}{3}} \sin\frac{n\pi x}{3}\Big|_0^3 - \frac{2}{3}\left(x \cdot \frac{1}{\frac{n\pi}{3}} \sin\frac{n\pi x}{3}\Big|_0^3 - \int_0^3 \frac{1}{\frac{n\pi}{3}} \sin\frac{n\pi x}{3}\,dx \right)$$

$$= -\frac{2}{3} \cdot \left(\frac{3}{n\pi}\right)^2 \cos\frac{n\pi x}{3}\Big|_0^3 = -\frac{2}{3}\left(\frac{3}{n\pi}\right)^2 \left((-1)^n - 1\right)$$

For odd extension $a_n = 0$

$$b_n = \frac{2}{3}\int_0^3 (3-x)\sin\frac{n\pi x}{3}dx = \frac{2}{3}\int_0^3 3\sin\frac{n\pi x}{3}dx - \frac{2}{3}\int_0^3 x\sin\frac{n\pi x}{3}dx$$

$$= -\frac{2}{\frac{n\pi}{3}}\cos\frac{n\pi x}{3}\Big|_0^3 - \frac{2}{3}\left(x\cdot\left(-\frac{1}{\frac{n\pi}{3}}\cos\frac{n\pi x}{3}\right)\Big|_0^3 - \int_0^3 (x)'\left(-\frac{1}{\frac{n\pi}{3}}\cos\frac{n\pi x}{3}\right)dx\right)$$

$$= -\frac{2}{\frac{n\pi}{3}}\left((-1)^n - 1\right) + \frac{2}{3}\left(+\frac{3}{n\pi}\right)\cdot 3\cdot(-1)^n + \frac{2}{3}\int_0^3 -\frac{1}{\frac{n\pi}{3}}\cos\frac{n\pi x}{3}dx$$

$$b_n = -\frac{6}{n\pi}\left((-1)^n - 1\right) + \frac{6}{n\pi}(-1)^n = \frac{6}{n\pi}$$