1 Leaking Bucket

This engineering application may be used to augment the materials in §2.3: Modeling with First Order Equations. This application is problem # 6 in §2.3.

The leaking bucket in figure 1 can be described by investigating the water level $h(t)$ as a function of time. The volume conservation of water in the system is represented by the balance of volumetric flow rate $Q$ as follows:

$$Q_{in} - Q_{out} = Q_{stored}. \tag{1}$$

In the case when no water is flowing into the tank, $Q_{in} = 0$, we obtain

$$Q_{stored} = -Q_{out}. \tag{2}$$

The volumetric flow rate $Q_{stored}$ can be calculated by multiplying the velocity by the area of the tank

$$Q_{stored} = A_{tank} \frac{dh(t)}{dt}. \tag{3}$$

$Q_{out}$ is computed by multiplying the flow velocity by the area of the spout $A_{spout}$

$$Q_{out} = A_{spout}v(t). \tag{4}$$

![Figure 1: Sketch of a leaking bucket.](image-url)
where \( v(t) \) is the velocity of water coming out of the straw. For fluids of height \( h(t) \), the velocity of water coming out at the bottom is \( v(t) = \sqrt{2gh(t)} \). Therefore we arrive at the governing equation for \( h(t) \) as

\[
A_{tank} \frac{dh(t)}{dt} = -A_{spout} \sqrt{2gh}.
\] (5)

Rearranging terms, we obtain the following equation

\[
\frac{dh(t)}{dt} = -K \sqrt{h},
\] (6)

with \( K = \frac{A_{spout}}{A_{tank}} \sqrt{2g} > 0 \). Before solving equation 6, we observe that the water height is decreasing with time as \( \frac{dh}{dt} < 0 \) for all \( h \geq 0 \). With the initial condition \( h(0) = h_0 \), equation 6 can be solved by separation of variables as follows.

\[
\frac{dh}{\sqrt{h}} = -K dt, \quad \Rightarrow \quad \frac{dh}{\sqrt{h}} = -K dt.
\] (7)

Integrating both sides

\[
\int \frac{dh}{\sqrt{h}} = \int -K dt, \quad \Rightarrow \quad 2\sqrt{h} = -Kt + c,
\] (8)

where the integration constant \( c = 2\sqrt{h_0} \). Thus the water height can be expressed explicitly in terms of time as

\[
h(t) = \left( \sqrt{h_0} - \frac{Kt}{2} \right)^2.
\] (9)

Note that the solution \( h(t) \) in equation 9 decreases from the initial height \( h_0 \), and at time \( t_{end} = \frac{2\sqrt{h_0}}{K} \), the water is completely drained out (by gravity) and \( h(t_{end}) = 0 \).

2 Forced Vibrations
1) WITH DAMPING \[ m\ddot{u} + f u' + ku = F_0 \cos \omega t \] \hspace{1cm} (8) \hspace{1cm} \text{damping} \hspace{1cm} \to 0 \text{ damping} \hspace{1cm} F_0 \cos \omega t \text{ 'harmonic' forcing} \\

Solution \[ u(t) = u_c(t) + u_p(t) \]

\text{particular solution} \[ u_p = F_0 \cos \omega t \]

\text{complementary solution} \[ u_c = 0 \]

From p. 37, \[ u_c(t) = \frac{A e^{\alpha t} + B e^{-\alpha t}}{\alpha^2 - 4\Omega^2} \]

\[ t^2 - 4\Omega^2 > 0, \quad \gamma, \Omega < 0 \]

\[ t^2 - 4\Omega^2 = 0, \quad \gamma = \Omega = \frac{1}{2\mu} \]

\[ t^2 - 4\Omega^2 < 0, \quad \mu = \sqrt{4\mu^2 - \Omega^2} \]

with damping, \( t \to \infty, \) \( u_c(t) \to 0 \) as \( t \to \infty \). So, \( u_c(t) \), which contains the initial condition data \( u(0) \) and \( u'(0) \) in \( A, B, \alpha, \) and \( \Omega \), is 'transient' - negligible for large times.

\[ u_p(t) = A \cos \omega t + B \sin \omega t \]

\[ \text{periodic, steady-state (forced)} \]

\[ = R \cos(\omega t - \delta) \] \hspace{1cm} (10) \hspace{1cm} \text{large-time response.} 

Find \( R, \delta \) by substitution in (8). \[ (u_p' = -R \sin(\omega t - \delta), \ u_p'' = -R^2 \cos(\omega t - \delta)) \]

\[ R \left[ -m \Omega^2 \cos(\omega t - \delta) - f \Omega \sin(\omega t - \delta) + k \cos(\omega t - \delta) \right] = F_0 \cos \omega t. \]

\[ \therefore \frac{k}{F_0} \left( \{ k - m \Omega^2 \} \cos \delta + f \Omega \sin \delta + k \cos \omega t \right) = \cos \omega t. \]

\[ \text{equate (lin. indep.) \ \sin \omega t, \ \cos \omega t \ terms, \ \text{put} \ \omega \delta = \frac{k}{m} \ \text{to eliminate} \ \delta.} \]
\[ \sin \theta = \mu (w_0^2 - w^2) \sin \theta - F_0 \cos \theta = 0 \]  \hspace{1cm} (i) \\
\cos \theta = \frac{F_0}{K} \hspace{1cm} (ii) \\

Solve \( \mu (w_0^2 - w^2) (i) + F_0 (ii) \Rightarrow \)

\[ \left( \mu^2 (w_0^2 - w^2)^2 + F_0^2 \right) \sin \theta = \frac{F_0}{K} \]

\[ \mu (w_0^2 - w^2) (i) - F_0 (i) \Rightarrow \]

\[ \left( \mu^2 (w_0^2 - w^2)^2 + F_0^2 \right) \cos \theta = \frac{F_0 \mu (w_0^2 - w^2)}{K} \]

Let \( \Delta = \sqrt{\left( \mu^2 (w_0^2 - w^2)^2 + F_0^2 \right)} \). Square and add \( \Rightarrow \)

\[ \Delta^2 (\cos^2 \theta + \sin^2 \theta) = \left( \frac{F_0}{K} \right)^2 \cdot \Delta^2 \]

\[ \therefore \Delta = \frac{F_0}{K} \text{, then } \cos \theta = \frac{\mu (w_0^2 - w^2)}{\Delta}, \text{ } \sin \theta = \frac{F_0}{\Delta} \]  \hspace{1cm} (11)

\[ \exp \left( \Delta \cos \left( \omega t - \phi \right) \right) \]

Amplitude: \( \quad \text{Phase Difference, Between Response } \phi \text{ and Forcing } \omega \).

How do \( \Delta \) and \( \phi \) depend on \( w_0 \) ? \( \Delta = \sqrt{\left( \mu^2 (w_0^2 - w^2)^2 + F_0^2 \right)} \) \hspace{1cm} (12)

\( \omega_0^2 = \frac{K}{m} \)

\[ \frac{\Delta k}{F_0} = \frac{1}{\sqrt{\left( 1 - \frac{w^2}{\omega_0^2} \right)^2 + \left( \frac{\omega^2}{\omega_0^2} \right)} \hspace{1cm} (\text{13}) \]

where \( \Gamma = \frac{\omega^2}{\omega_0^2} \).

Here, each 'group': \( \frac{\Delta k}{F_0}, \frac{\omega^2}{\omega_0^2}, \omega \) is dimensionless. Parameters (\( w, k, \)

\( F_0, \omega_0 \)) to 3 dimensionless Groups.

\( k \to 0 \); \( \frac{\Delta k}{F_0} \to 1 \); \( \omega \to \omega_0 \); \( k \to 0 \); \( \frac{\omega^2}{\omega_0^2} \to 0 \)

LIMIT OF \( \omega \) \( \text{High freq. forcing. No Amplitude.} \)
At what forcing frequency ω is the amplitude a maximum?

\[ Q' = \left( \frac{\Delta k}{\omega_0} \right)^2 = \frac{1}{(1-\rho)^2 + \rho} \quad \text{where} \quad \rho = \frac{\omega^2}{\omega_0^2} \quad \frac{d}{d\rho} \left( (1-\rho)^2 + \rho \right) = 0 \Rightarrow \]

\[-2(1-\rho) + \rho = 0 \Rightarrow \rho = \frac{\omega^2}{\omega_0^2} = 1 - \frac{1}{2} \Rightarrow \omega_m^2 = \omega_0^2 \left( 1 - \frac{1}{2} \right) \left( \frac{\omega_0^2}{\omega_m^2} \right) \]

or \[ \omega_m^2 = \omega_0^2 \left( 1 - \frac{\omega^2}{\omega_0^2} \right) \] (This causes max. amplitude)

Then \[ Q' = \frac{1}{\Delta(1-\rho)} \Rightarrow \frac{\Delta k}{\omega_0} = \frac{k}{\Delta \omega_0 \left( 1 - \frac{\omega^2}{4\omega_0^2} \right)^{1/2}}. \]

For small damping, \( t << \omega_0 \Delta \omega \) = \( \frac{k}{\omega_0} \), \( \frac{\Delta k}{\omega_0} \) ~ \( \frac{k}{\Delta \omega_0} \) \( \rightarrow \) \( \infty \) as \( t \rightarrow 0 \).

\[ \frac{\Delta k}{\omega_0} \]
\[ \omega_0 \]

For \( \Delta = 0.015 \)

Max. and \( \frac{\Delta k}{\omega_0} > 8 \)

"Near Resonance"

(\( \text{This is Fig. 382, p. 321 in book} \))

**ZERO DAMPING:** \( \Delta = 0 \Rightarrow \rho = 0 \), Then (13) \( \Rightarrow \frac{\Delta k}{\omega_0} = \frac{1}{1 - \frac{\omega^2}{\omega_0^2}} \)

\( \rightarrow \infty \) as \( \omega \rightarrow \omega_0 \)

(ii) \( \delta \)-phase shift formula (11) \( \sin \delta = \frac{\omega_1 A}{\Delta} \), \( \cos \delta = \frac{\omega_0 (\omega_1^2 - \omega_0^2)}{\Delta} \), \( \Delta = \sqrt{\left( \omega_1^2 - \omega_0^2 \right)^2 + \omega_0^2 \omega_1^2} \)

\( \omega \rightarrow 0 \): \( \sin \delta \rightarrow 0 \), \( \cos \delta \rightarrow 1 \) \( \Rightarrow \delta \rightarrow 0 \). Then (low frequency forcing) response \( \omega_p = A \cos (\omega t - \delta) \) nearly in phase with forcing \( \omega_0 \cos \omega_0 t \).

\( \frac{\omega}{\omega_0} = 1 \); \( \sin \delta = 1 \), \( \cos \delta = 0 \) \( \Rightarrow \delta = \frac{\pi}{2} \).

Response is behind forcing by \( \frac{\pi}{2} \)
\[
\frac{c_l}{c_o} \gg 1: \quad (\frac{c_l \to \infty}{c_o}) \quad \text{sin} \delta \to \frac{\delta}{\omega} \to 0, \quad \text{cos} \delta \to -1 \to \delta \to \pi.
\]

**Response Lags Forcing by \pi**

"\pi out of phase."

\[
\delta
\]

\[
\pi
\]

\[
\begin{array}{c}
\text{Response Lags Forcing by } \pi \\
\text{"\pi out of Phase"}
\end{array}
\]

\[
\text{small damping} \\
\left( \frac{c_l}{m} \ll 1 \right)
\]

\[
\text{rapid change in } \delta
\]

\[
\text{from } \delta = \pi \text{ for } c_l = c_o
\]

\[
\text{(This is Fig. 3.3.3, p. 212 in book)}
\]

\[
\frac{c_l}{c_o}
\]

\[
\text{2) Forces, with no damping, (p. 214)}
\]

\[
\text{not resonant.}
\]

\[
\begin{aligned}
\text{General solution:} \\
\dot{u} + \omega_0 u &= F_0 \cos \omega t \\
(17) \\
\omega_0 &= \sqrt{\frac{c_l}{m}} \\
\end{aligned}
\]

\[
\begin{aligned}
\text{Check this.}
\end{aligned}
\]

\[
\text{up check this.}
\]

\[
\text{No forcing periodic}
\]

\[
\text{no transient}
\]

\[
\text{and substitute for}
\]

\[
\text{unresonant coefficients.}
\]

\[
\text{c}_1 \text{, and } c_2 \text{ determined by IC's. Take example of motion beginning from zero displacement (} u(0) = 0 \text{) at rest (} u'(0) = 0 \text{) — only the forcing excites: } u(0) = 0 \Rightarrow \}
\]

\[
\begin{aligned}
\text{c}_1 &= -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{, } \\
u(0) = 0 \Rightarrow \quad c_2 &= 0 \quad \text{Then}
\end{aligned}
\]

\[
\begin{aligned}
\dot{u} &= \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \\
(20)
\end{aligned}
\]

\[
\text{Equal Amplitudes}
\]

\[
\text{Recall } \cos(A-B) = \cos(A+B) = 2 \sin A \cdot \sin B.
\]

\[
\text{but } A-B = \omega t \quad \Rightarrow \quad A = (\omega + \omega_0) t, \quad B = (\omega - \omega_0) t.
\]
\[ u = \frac{2u_0}{\omega (\omega^2 - \omega_0^2)} \sin(\omega_0 - \omega)t \sin(\omega_0 + \omega)t \] (21)

For \( \omega_0 \) close to \( \omega_0 \)
But \( \omega_0 \neq \omega_0 \).
Near resonance:
No Banzans.

\[ \omega_0 + \omega \approx \omega_0 \]

[Amplitude modulation]
BEATS

\[ \phi(t) = (2.77...) \sin \frac{\pi}{10} \sin \frac{9\pi}{10} \]

3) Forces, no Banzans
Resonant (p215)

\[ \omega = \omega_0 \]
Full response for
\[ u_p = (A + B) \cos \omega_0 t + (C + D) \sin \omega_0 t \]

\[ \omega_0 = \frac{1}{2 \omega_0} \]

The general solution is
\[ u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{\phi(t)}{2 \omega_0} \sin \omega_0 t \]

The ICs give \( c_1, c_2 \)

\[ u_p \text{ oscillatory and 'diodes without bound'} \]
For large \( t \), \( \text{hierarchy/swallow amplitude model 'fang' of } u \)