MATH 222 EXAM II, Mar. 12, 2014

Read each problem carefully. Show all your work for each problem. No Calculators!

1. (12) Find the general solutions of the following differential equations

$$\text{(a) } \frac{1}{\pi}y''-2y'+\pi y=0, \qquad \text{(b) } 2y''+2y'+3y=0.$$

2. (14) Determine a suitable form for the particular solution if the method of undetermined coefficients is to be used. DO NOT solve for the coefficients.

$$\text{(a) } y''+4y=(7t+1)\,e^{-5t}\cos 3t+t\sin 2t, \qquad \text{(b) } y''+2y'+y=3t\,e^{-t}.$$

3. (14) Solve the following initial value problem

$$y''-y'-2y=2e^{-t}, \quad y(0)=5, \quad y'(0)=\frac{1}{3}.$$

4. (15) Given that $y_1(t)=t$ is the solution of the following differential equation

$$t^2y''-t(t+2)y'+(t+2)y=0, \quad t>0,$$

find a second solution using the method of reduction of order.

5. (15) Use **the method of variation of parameters** to find the particular solution of the following differential equation

$$y''+4y=3\cos 2t.$$

Notice that $\cos 4t=2\cos^2 2t-1, \quad \sin 4t=2\cos 2t\sin 2t$.

6. A steel ball weighing 16 lb is attached to a vertical spring, stretching the spring 6 inches from its natural length. Then the ball is pulled down 2 inches below the equilibrium position and released. Assuming that the initial velocity is zero and the string is attached to a damper with damping coefficient γ lb sec/ft. The gravitational acceleration is 32ft/sec^2.

   (a) (10) Formulate the initial value problem for the problem described above.
   (b) (5) Determine γ such that the ratio of the quasi-frequency to the natural frequency is $\sqrt{63/64}$.

7. Seek power series solution of $(x^2+3)y''-7xy'+16y=0$ about x=0.

   (a) (10) Find the recurrence relation.
   (b) (5) Find the first two terms in each of the solutions y_1 and y_2 which form the fundamental set of solutions.

P.1

Solutions to Exam II, M222, March 2014

Prob 1 (a) $y'' - 2\pi y' + \pi^2 y = 0$

$r^2 - 2\pi r + \pi^2 = 0$

$(r-\pi)^2 = 0, \ r = \pi, \pi$

$\boxed{y = c_1 e^{\pi t} + c_2 t e^{\pi t}}$

(b) $2y'' + 2y' + 3y = 0$

$2r^2 + 2r + 3 = 0$

$r = \frac{-2 \pm \sqrt{2^2 - 4\cdot 2\cdot 3}}{2\cdot 2} = \frac{-2 \pm \sqrt{20}\,i}{4} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} i$

$\boxed{y = e^{-\frac{t}{2}} \left(c_1 \cos \frac{\sqrt{5}}{2} t + c_2 \sin \frac{\sqrt{5}}{2} t \right)}$

Prob 2 (a) $y'' + 4y = (7t+1) e^{-5t} \cos 3t + t \sin 2t$

corresponding homogeneous equation $y'' + 4y = 0$

$y_1 = \cos 2t, \ y_2 = \sin 2t$

$$\boxed{\begin{aligned} Y = {} & (At+B) e^{-5t} \cdot (C \cos 3t + D \sin 3t) \\ & + t(Et+F)(G \cos 2t + H \sin 2t) \end{aligned}}$$

(b) $y'' + 2y' + y = 3te^{-t}$

corresponding homogeneous equation $y'' + 2y' + y = 0$

$y_1 = e^{-t}, \ y_2 = te^{-t}$

$\boxed{Y = t^2 (At+B) e^{-t}}$

Prob 3

$y'' - y' - 2y = 2e^{-t}, \quad y(0) = 5, \quad y'(0) = \frac{1}{3}$

$y'' - y' - 2y = 0, \quad r^2 - r - 2 = 0, \quad r = 2, -1$

$y_1 = e^{+2t}, \quad y_2 = e^{-t}$

$Y = Ate^{-t}, \quad Y' = Ae^{-t} - Ate^{-t}, \quad Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

$Y'' - Y' - 2Y = (-2Ae^{-t} + Ate^{-t}) - (Ae^{-t} - Ate^{-t}) - 2Ate^{-t}$

$= -3Ae^{-t} = 2e^{-t}, \quad A = -\frac{2}{3}$

$y = c_1e^{2t} + c_2e^{-t} - \frac{2}{3}te^{-t}$

$y' = 2c_1e^{2t} - c_2e^{-t} - \frac{2}{3}e^{-t} + \frac{2}{3}te^{-t}$

$y(0) = c_1 + c_2 = 5$

$y'(0) = 2c_1 - c_2 - \frac{2}{3} = \frac{1}{3}$

$\begin{cases} c_1 + c_2 = 5 \\ 2c_1 - c_2 = 1 \end{cases} \quad c_1 = 2, \quad c_2 = 3$

$\boxed{y = 2e^{2t} + 3e^{-t} - \frac{2}{3}te^{-t}}$

Prob 4

$y_2 = vy_1, \quad y_2' = v'y_1 + vy_1', \quad y_2'' = v''y_1 + 2v'y_1' + vy_1''$

$t^2(v''y_1 + 2v'y_1' + vy_1'') - t(t+2)(v'y_1 + vy_1') + (t+2)vy_1 = 0$

$t^2(v''y_1 + 2v'y_1') - t(t+2)v'y_1 = 0$

$y_1 = t, \quad y_1' = 1, \quad y_1'' = 0$

$t^2(v''t + 2v'1) - t(t+2)v't = 0$

$t^3v'' + 2t^2v' - (t^3 + 2t^2)v' = 0$

$v'' - v' = 0, \quad v'' = v', \quad \ln v' = t, \quad v' = e^t, \; v = e^t$

$\boxed{y_2 = te^t}$

Prob 5

$y'' + 4y = 3\cos 2t$

$r^2 + 4 = 0$, $r = \pm 2i$, $y_1 = \cos 2t$, $y_2 = \sin 2t$

$Y = t\cdot(A\cos 2t + B\sin 2t)$

$Y' = A\cos 2t + B\sin 2t + 2t(-A\sin 2t + B\cos 2t)$

$Y'' = -2A\sin 2t + 2B\cos 2t + 2(-A\sin 2t + B\cos 2t)$
$\quad + 2t(-2A\cos 2t - 2B\sin 2t)$

$Y'' + 4Y = 3\cos 2t,$

$-4A\sin 2t + 4B\cos 2t - 4\cdot t(A\cos 2t + B\sin 2t) + 4t(A\cos 2t + B\sin 2t)$
$= 3\cos 2t$

$\cos 2t$: $4B - 4t\cdot A + 4t\cdot A = 3$ $\quad B = 3/4$

$\sin 2t$: $-4A - 4t\cdot B + 4t\, B = 0$ $\quad A = 0$

$Y = t\,\frac{3}{4}\sin 2t = \frac{3}{4}t\sin 2t$

variation of parameters: $\quad W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$

$Y = \left(\int \frac{-3\cos 2t\,\sin 2t}{2}\right)\cos 2t + \left(\int \frac{+3\cos 2t\,\cos 2t}{2}\right)\sin 2t$

$= -\frac{3}{4}\left(\int \sin 4t\right)\cos 2t + \frac{3}{2}\left(\int \frac{\cos 4t + 1}{2}\right)\sin 2t$

$= \frac{3}{16}\cos 4t\cdot\cos 2t + \frac{3}{4}\cdot\left(\frac{1}{4}\sin 4t + t\right)\sin 2t$

$= \frac{3}{16}\cos 2t + \frac{3}{4}t\sin 2t \implies \boxed{Y = \frac{3}{4}t\sin 2t}$

P.4

Prob 6 $\quad 16 = \frac{1}{2}\cdot k, \quad k = 32\ \frac{lb}{ft}$

$m = \frac{16}{32} = \frac{1}{2}$

$\frac{1}{2}u'' + \gamma u' + 32u = 0$

(a) $\quad u'' + 2\gamma u' + 64u = 0, \quad u(0) = \frac{2}{12}, \quad u'(0) = 0$

(b) $\quad r^2 + 2\gamma r + 64 = 0$

$(r+\gamma)^2 + 64 - \gamma^2 = 0, \quad (r+\gamma)^2 = \left(\sqrt{64-\gamma^2}\,i\right)^2$

$r = -\gamma \pm \sqrt{64-\gamma^2}\,i$

natural frequency $= 8$

quasi-frequency $= \sqrt{\frac{63}{64}}\,8 = \sqrt{63} = \sqrt{64-\gamma^2}, \quad \gamma^2 = 1, \quad \boxed{\gamma = 1}$ $\frac{lb\,sec}{ft}$

Prob 7 $\quad (x^2+3)y'' - 7xy' + 16y = 0$ about $x = 0$

$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$(x^2+3)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 7x\sum_{n=1}^{\infty} n a_n x^{n-1} + 16\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 3\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 7\sum_{n=1}^{\infty} n a_n x^n + 16\sum_{n=0}^{\infty} a_n x^n = 0$

x^0 : $\quad 3\cdot 2\cdot 1\cdot a_2 + 16a_0 = 0, \quad a_2 = -\frac{16}{6}a_0 = -\frac{8}{3}a_0$

x^1 : $\quad 3\cdot 3\cdot 2\cdot a_3 - 7\cdot a_1 + 16a_1 = 0, \quad a_3 = \frac{1}{2}a_1$

$n \geq 2$: $\quad n(n-1)a_n + 3(n+2)(n+1)a_{n+2} - 7na_n + 16a_n = 0$

$(n^2 - 8n + 16)a_n = -3(n+2)(n+1)a_{n+2}, \quad a_n = -\frac{3(n+2)(n+1)}{(n-4)^2}a_{n+2}$

P.5

(a) $a_{n+2} = -\dfrac{(n-4)^2}{3(n+2)(n+1)} a_n$

(b) $y_1 = a_0 - \frac{8}{3} a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots$

$y_2 = a_1 x + \frac{1}{2} a_1 x^3 + a_5 x^5 + a_7 x^7 + \cdots$