MATH 222 EXAM II

Oct. 21, 2015

Read each problem carefully. Show all your work for each problem. No Calculators!

1. (8 pts) Solve the following initial value problem for $y(t)$

$$4y'' + 12y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = -2.$$

2. (10 pts) Determine a suitable form for the particular solution $Y(t)$ if the method of undetermined coefficients is to be used. Do not solve for the coefficients.

   (a) $y'' + 9y = t^2 e^{-3t} - t\cos 3t,$ (b) $y'' + 6y' + 9y = (t^2 + 1)e^{-3t}.$

3. (20 pts) Consider the differential equation $t^2 y'' + 2ty' - 2y = g(t)$ for $t > 0$.

   (a) When $y_1(t) = t^{-2}$ is a solution of the differential equation with $g(t) = 0$, find the second linearly independent solution $y_2(t)$.

   (b) If $g(t) = 1$, find the particular solution $Y(t)$ using the method of variation of parameters.

4. (14 pts) Solve the following initial value problem:

$$y'' + 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$$

5. (14 pts) Find the general solution of the following differential equation

$$4y'' - 4y' + y = e^{\frac{t}{2}}\sqrt{t}.$$

6. A force of 1 N ($= 1$ Kg m/s^2) stretches a linear spring 0.1 meters. A mass of 1 Kg is hung from the spring and its motion is damped by a viscous damper that exerts a force of 2 N when the mass velocity is 1 m/s. Initially the mass is pushed up 0.5 m above its equilibrium position and then released from rest.

   (a) (15 pts) Find the displacement $u(t)$ of the mass at any time t.

   (b) (5 pts) Express the solution in the form of $u(t) = Re^{-\lambda t}\cos(\omega t - \delta)$.

7. Consider the mass-spring system described in Problem 6 without the viscous damper (but with the same mass, spring, and initial conditions).

   (a) (4 pts) What is the natural frequency ω_0 of the system?

   (b) (10 pts) When an external forcing $F(t) = \cos(t)$ is applied to the system, find the displacement $u(t)$.

Solution to Exam II, M222 Fall 2015

Prob 1: $4y'' + 12y' + 9y = 0,\quad y(0) = 2,\ y'(0) = -2$

$4r^2 + 12r + 9 = 0,\quad (2r+3)^2 = 0,\quad r = -\frac{3}{2}, -\frac{3}{2}$

$y(t) = c_1 e^{-\frac{3}{2}t} + c_2 t e^{-\frac{3}{2}t}$

$y'(t) = -\frac{3}{2} c_1 e^{-\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t} + c_2 \cdot (-\frac{3}{2}) t e^{-\frac{3}{2}t}$

$y(0) = c_1 = 2$

$y'(0) = -\frac{3}{2}c_1 + c_2 = -2,\quad c_2 = \frac{3}{2}c_1 - 2 = 1$

$$\boxed{y(t) = 2e^{-\frac{3}{2}t} + te^{-\frac{3}{2}t}}$$

Prob 2 (a) $y'' + 9y = t^2 e^{-3t} - t\cos 3t$

$r^2 + 9 = 0,\quad r = \pm 3i,\quad y_h = c_1\cos 3t + c_2 \sin 3t$

$Y = \cancel{c_1\cos 3t + c_2\sin 3t +}\ e^{-3t}(At^2 + Bt + C)$
$\quad + t(Dt+E)\cos 3t + t(Ft+G)\sin 3t$

(b) $y'' + 6y' + 9y = (t^2+1)e^{-3t}$

$r^2 + 6r + 9 = 0,\quad (r+3)^2 = 0,\quad r = -3, -3,\quad y_h = c_1 e^{-3t} + c_2 t e^{-3t}$

$Y = t^2(At^2 + Bt + c)e^{-3t}$

Prob 3: $t^2 y'' + 2ty' - 2y = g(t)$ for $t > 0$.

(a) $y_2 = v \cdot t^{-2},\quad y_2' = -2t^{-3}\cdot v + v' t^{-2}$

P.2

$y_2'' = 6\cancel{t}^{-4}v - 2t^{-3}v' - 2t^{-3}v' + v''t^{-2}$

$= v'' t^{-2} - 4t^{-3}v' + 6t^{-4}v$

$t^2 y_2'' + 2t y_2' - 2y_2 = 0$

$t^2\left(\frac{v''}{t^2} - \frac{4v'}{t^3} + \frac{6v}{t^4}\right) + 2t\left(-\frac{2v}{t^3} + \frac{v'}{t^2}\right) - 2\cdot\frac{v}{t^2} = 0$

$v'' - \frac{4}{t}v' + \frac{6v}{t^2} + \frac{2v'}{t} - \frac{4v}{t^2} - 2\frac{v}{t^2} = 0$

V terms: $\frac{6v}{t^2} - \frac{4v}{t^2} - \frac{2v}{t^2} = 0$ ✓

$v'' - \frac{2}{t}v' = 0, \quad \frac{v''}{v'} = \frac{2}{t}, \quad \ln v' = 2\ln t, \quad v' = t^2$

$v = \frac{t^3}{3}, \quad y_2 = \frac{t^3}{3}\cdot t^{-2} = \frac{t}{3}$

⇒ second linearly independent solution $\boxed{y_2 = t \text{ or } \frac{t}{3}}$

(b) $t^2y'' + 2ty' - 2y = 1, \quad y'' + \frac{2t}{t^2}y' - \frac{2}{t^2}y = \frac{1}{t^2}$

$W(y_1, y_2) = \begin{vmatrix} t^{-2} & t \\ -2t^{-3} & 1 \end{vmatrix} = t^{-2} + 2t^{-2} = 3t^{-2}$

$Y(t) = \left(-\int \frac{g(t)\,y_2(t)}{W}\,dt\right) y_1 + \left(\int \frac{g(t)\,y_1(t)}{W}\,dt\right) y_2$

$\int \frac{g(t)y_2}{W} = \int \frac{\frac{1}{t^2}\cdot t}{3t^{-2}}\,dt = \int \frac{t}{3}\,dt = \frac{1}{6}t^2$

$\int \frac{g(t)y_1}{W} = \int \frac{\frac{1}{t^2}\cdot\frac{1}{t^2}}{3t^{-2}} = \int \frac{1}{3t^2} = -\frac{1}{3}t^{-1}$

P.3.

$$\boxed{Y = -\frac{1}{6}t^2 \cdot t^{-2} + \left(-\frac{1}{3}\frac{1}{t}\right)t = -\frac{1}{6} - \frac{1}{3} = -\frac{1+2}{6} = -\frac{1}{2}}$$

Prob 4.

$y'' + 2y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 0$

$y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$,

$(r+1)^2 = -1$, $r = -1 \pm i$ $\quad y_c = (c_1 \cos t + c_2 \sin t)e^{-t}$

$Y = Ae^{-t}$, $\quad Y' = -Ae^{-t}$, $\quad Y'' = Ae^{-t}$

$(A - 2A + 2A)e^{-t} = e^{-t}$, $\quad A = 1$

$y = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t}$

$y' = -e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t}(-c_1 \sin t + c_2 \cos t) - e^{-t}$

$y(0) = 0 = c_1 + 1$, $\quad c_1 = -1$

$y'(0) = 0 = -1 \cdot (-1) + 1 \cdot c_2 - 1$, $\quad c_2 = 0$.

$$\boxed{y(t) = -e^{-t}\cos t + e^{-t}}$$

Prob 5.

$4y'' - 4y' + y = e^{t/2}\sqrt{t}$

$(2r-1)^2 = 0$, $r = \frac{1}{2}, \frac{1}{2}$

$y_h(t) = c_1 e^{t/2} + c_2 t e^{t/2}$, $\quad y_1 = e^{t/2}$, $\quad y_2 = te^{t/2}$

$$Y = \left(-\int \frac{te^{t/2} \cdot \frac{e^{t/2}\sqrt{t}}{4}}{W}\right)e^{t/2} + \left(\int \frac{e^{t/2} \cdot \frac{e^{t/2}\sqrt{t}}{4}}{W}\right)te^{t/2}$$

$$W = \begin{vmatrix} e^{t/2} & te^{t/2} \\ \frac{1}{2}e^{t/2} & e^{t/2} + \frac{1}{2}te^{t/2} \end{vmatrix} = e^{t}$$

P.4

$$\int \frac{\frac{t^{3/2}}{4} e^t}{e^t} dt = \frac{1}{4} \int t^{3/2} dt = \frac{1}{4} \cdot \frac{1}{5/2} t^{5/2}$$

$$\int \frac{\frac{t^{1/2}}{4} e^t}{e^t} dt = \frac{1}{4} \int t^{1/2} dt = \frac{1}{4} \frac{1}{3/2} t^{3/2}$$

$$Y = -\frac{1}{10} t^{5/2} \cdot e^{t/2} + \frac{1}{6} t^{5/2} e^{t/2} = \frac{-3+5}{30} t^{5/2} e^{t/2}$$

$$Y = \frac{1}{15} t^{5/2} e^{t/2}$$

$$y = C_1 e^{t/2} + C_2 t e^{t/2} + \frac{1}{15} t^{5/2} e^{t/2}$$

Prob 6: $k = \frac{1}{0.1} = 10$

$\gamma = \frac{2}{1} = 2$

$m = 1$

$u'' + 2u' + 10u = 0$, $u(0) = -0.5$, $u'(0) = 0$

(a) $r^2 + 2r + 10 = 0$,

$(r+1)^2 = -9$, $r = -1 \pm 3i$

$u = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$

$u'(t) = -e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$
$\quad + e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$

$u(0) = -\frac{1}{2} = c_1$ $\qquad$ $c_1 = -\frac{1}{2}$, $c_2 = \frac{c_1}{3} = -\frac{1}{6}$

$u'(0) = -c_1 + 3c_2 = 0$

$$u(t) = \left(-\frac{1}{2}\cos 3t - \frac{1}{6}\sin 3t\right) e^{-t}$$

P. 65

(b) $-\frac{1}{2}\cos 3t - \frac{1}{6}\sin 3t = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{6}\right)^2} \cdot \left(\frac{\frac{1}{2}}{\sqrt{\frac{10}{36}}} \cos 3t + \frac{\frac{1}{6}}{\sqrt{\frac{10}{36}}} \sin 3t \right)$

$\Downarrow$

$R = \sqrt{\frac{1}{2^2} + \frac{1}{36}} = \frac{\sqrt{10}}{6}$

$u(t) = -R \cdot e^{-t} \cdot \cos(3t - \delta), \quad \delta = \tan^{-1}\left(\frac{\frac{1}{6}}{\frac{1}{2}}\right) = \tan^{-1}\left(\frac{1}{3}\right)$

Prob 7: (a) $u'' + 10u = \cos t$

natural freq. $\omega_0 = \sqrt{10}$

(b) $u(0) = -0.5, \quad u'(0) = 0$

$u = A\cos\sqrt{10}t + B\sin\sqrt{10}t + A\cos t + B\sin t$

$(-A + 10A) = 1 \qquad A = \frac{1}{9}, \qquad B = 0$

$u = A\cos\sqrt{10}t + B\sin\sqrt{10}t + \frac{1}{9}\cos t$

$u' = \sqrt{10}\,(-A\sin\sqrt{10}t + B\cos\sqrt{10}t) - \frac{1}{9}\sin t$

$u(0) = 0.5 \quad \cancel{A = \frac{1}{9}}$

$A + \frac{1}{9} = -\frac{1}{2}, \quad A = -\frac{1}{2} - \frac{1}{9} = -\frac{11}{18}$

$u'(0) = 0, \quad B \cdot \sqrt{10} = 0, \quad B = 0$

$$\boxed{u(t) = -\frac{11}{18}\cos\sqrt{10}t + \frac{1}{9}\cos t}$$