MATH 222 EXAM III, Apr. 23, 2014

Read each problem carefully. Show all your work for each problem. No Calculators!

1. (14) Solve the initial value problem:

$$x^2y''+xy'+y=0, \quad y(1)=1, \quad y'(1)=1$$

2. (a) (7) Write the following equation as a system of first-order differential equations

$$y'''-3y''+8y=7\cos t+9.$$

(b) (5) Write the system in the form of $\boldsymbol{x}'=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}$, where $\boldsymbol{A}$ is a matrix and $\boldsymbol{x}(t)$ and $\boldsymbol{b}(t)$ are vectors.

3. (a) (10) Express the following function in terms of unit step functions

$$f(t)=\begin{cases} t & 0\le t<1 \\ 1 & 1\le t<3 \\ t & 3\le t \end{cases}.$$

(b) (5) Find its Laplace transform.

4. (16) Find the inverse Laplace transform of

(a) $F(s)=\dfrac{2s-3}{s^2+2s+5}$, (b) $F(s)=\dfrac{e^{-2s}(s-3)}{(s-1)^4}$.

5. (15) Solve the following initial value problem

$$y''+2y'-3y=\delta(t-3), \quad y(0)=0, \quad y'(0)=\pi.$$

6. (14) Express the solution of the following initial value problem in terms of a convolution integral

$$y''+4y=\sin 3t+\cos t, \quad y(0)=0, \quad y'(0)=0.$$

7. Find the Laplace transform of

(a) (7) $f(t)=t\sin t$,

(b) (7) $f(t)=\int_0^t y^2\, u_4(t-y)\, e^{3y}\, dy$.

Prob 1: $x^2y'' + xy' + y = 0, \quad y(1) = 1, \; y'(1) = 1$

$y = x^r, \quad y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2}$

$r(r-1)x^r + rx^r + x^r = 0$

$x^r[r(r-1) + r + 1] = 0 \qquad r^2 + 1 = 0, \qquad r = \pm i$

$y_1 = \cos(\ln x), \qquad y_2 = \sin(\ln x)$

$y = ay_1 + by_2 = a\cos\ln x + b\sin\ln x$

$y' = -a \cdot \sin(\ln x) \cdot \frac{1}{x} + b \cdot \cos(\ln x) \cdot \frac{1}{x}$

$\left.\begin{aligned} y(1) &= 1 = a \\ y'(1) &= 1 = b \end{aligned}\right\} \qquad y = \cos(\ln x) + \sin(\ln x)$

Prob 3: (a) $f(t) = a + b \cdot u_1 + c\,u_3$

$a = t$

$a + b = 1 \qquad b = 1 - a = 1 - t,$

$a + b + c = t \qquad c = t - b - a = t - (1-t) - t$

$= +t - 1$

$f(t) = t + (1-t)u_1 + (+t-1)u_3$

(b) $\mathcal{L}[f(t)] = \frac{1}{s} - \frac{e^{-s}}{s^2} + \mathcal{L}[(t-3)u_3 + 2u_3]$

$= \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s^2} + 2 \cdot \frac{e^{-3s}}{s}$

Prob 4: (a) $F(s) = \frac{2s-3}{s^2+2s+5}$

$$\mathcal{L}^{-1}[F] = \mathcal{L}^{-1}\left[\frac{2s-3}{(s+1)^2+2^2}\right] = \mathcal{L}^{-1}\left[\frac{2(s+1)-5}{(s+1)^2+2^2}\right]$$

$$= \mathcal{L}^{-1}\left[2 \cdot \frac{s+1}{(s+1)^2+2^2}\right] - \mathcal{L}^{-1}\left[\frac{5}{2} \cdot \frac{2}{(s+1)^2+2^2}\right]$$

$$= 2 \cdot e^{-t} \cdot \cos 2t - \frac{5}{2} \cdot e^{-t} \sin 2t$$

(b) $F(s) = \frac{e^{-2s}(s-3)}{(s-1)^4}$

$$\mathcal{L}^{-1}[F] = \mathcal{L}^{-1}\left[\frac{s-3}{(s-1)^4} e^{-2s}\right] = \mathcal{L}^{-1}\left[\left[\frac{1}{(s-1)^3} - \frac{2}{(s-1)^4}\right] e^{-2s}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)^3}\right] = \frac{1}{2} \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) = \frac{1}{2} G^{(2)}(s) \quad \text{where } G(s) = \frac{1}{s-1}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{2} G^{(2)}(s)\right] = \frac{1}{2} (-t)^2 \cdot e^{t}$$

$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)^4}\right] = \frac{2}{-6} \frac{d^3}{ds^3}\left(\frac{1}{s-1}\right) = -\frac{2}{6} \cdot H^{(3)}(s) \quad \text{where } H(s) = \frac{1}{s-1}$$

$$\therefore \mathcal{L}^{-1}\left[-\frac{1}{3} H^{(3)}(s)\right] = -\frac{1}{3} (-t)^3 e^{t}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{e^{-2s}(s-3)}{(s-1)^4}\right] = \frac{1}{2}(-t+2)^2 e^{t-2} U_2$$

$$+ \frac{1}{3} (-t+2)^3 e^{t-2} U_2$$

Prob 5. $y''+2y'-3y=\delta(t-3),\quad y(0)=0,\quad y'(0)=\pi$

$$\mathcal{L}[y''+2y'-3y]=\mathcal{L}[\delta(t-3)]=e^{-3s}$$
$$s^2Y-sy(0)-y'(0)+2(sY-y(0))-3Y=e^{-3s}$$

$$(s^2+2s-3)Y-\pi=e^{-3s}$$

$$(s^2+2s-3)Y=\pi+e^{-3s}$$

$$Y=\frac{\pi}{s^2+2s-3}+\frac{e^{-3s}}{s^2+2s-3}=\frac{(\pi+e^{-3s})}{(s+3)(s-1)}=(\pi+e^{-3s})\left(\frac{A}{s+3}+\frac{B}{s-1}\right)$$

$$A(s-1)+B(s+3)=1$$
$$\begin{cases}A+B=0 \\ -A+3B=1\end{cases}\qquad \begin{matrix}A=-B,\ A=-\frac{1}{4} \\ 4B=1,\ B=\frac{1}{4}\end{matrix}$$

$$Y=-\frac{1}{4}\left(\frac{1}{s+3}-\frac{1}{s-1}\right)(\pi+e^{-3s})$$

$$\mathcal{L}[Y]=-\frac{\pi}{4}(e^{-3t}-e^{t})-\frac{1}{4}u_3\left(e^{-3(t-3)}-e^{+(t-3)}\right)$$

Prob 6. $y''+4y=\sin 3t+\cos t,\quad y(0)=0,\quad y'(0)=0$

$$\mathcal{L}[y''+4y]=\mathcal{L}[\sin 3t+\cos t]$$
$$(s^2+4)Y=\frac{3}{s^2+3^2}+\frac{s}{s^2+1^2}$$

$$Y=\frac{3}{(s^2+2^2)(s^2+3^2)}+\frac{s}{(s^2+1)(s^2+2^2)}$$

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$$y(t) = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{3}{(s^2+2^2)(s^2+3^2)} + \frac{s}{(s^2+1)(s^2+2^2)}\right]$$

$$= \int_0^t \sin 3(t-\tau)\,\frac{\sin 2\tau}{2}\,d\tau + \int_0^t \cos(t-\tau)\cdot\frac{\sin 2\tau}{2}\,d\tau$$

Prob 7: (a) $\mathcal{L}[t\sin t] = -F^{(1)}(s)$ where $F(s) = \mathcal{L}[\sin t] = \frac{1}{s^2+1}$

$$= \frac{1\cdot 2s}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2}$$

(b) $\mathcal{L}\left[\int_0^t y^2\cdot u_4(t-y)e^{3y}\,dy\right]$

$$= \mathcal{L}[t^2 e^{3t}]\cdot\mathcal{L}[u_4(t)]$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)\cdot e^{-4s} = \frac{2}{(s-3)^2}e^{-4s}$$

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