MATH 222 EXAM III

Nov. 18, 2015

Read each problem carefully. Show all your work for each problem. No Calculators!

1. (18 pts) Seek a power series solution of form $y = \sum_{n=0}^{\infty} a_n x^n$ for the following equation

$$y'' + xy' + 2y = 0.$$

   (a) Find the recurrence relation.

   (b) Find the first 3 terms of each of two linearly-independent solutions, y_1 and y_2.

2. (14 pts) Solve the following initial value problem

$$x^2y'' - 3xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 0.$$

3. (14 pts) Sketch the function

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 2 + t & 1 \le t < 2 \\ 5 & t \ge 2 \end{cases}$$

   and find its Laplace transform using only the definition of the Laplace Transform.

4. (16 pts) Find the Inverse Laplace transform of

$$(a) \;\; F(s) = \frac{e^{-3s}(s-2)}{s^2 + 4s + 8}, \qquad (b) \;\; F(s) = \frac{s^2 - 4s + 6}{(s-2)^3}$$

5. (13 pts) Solve the following initial value problem:

$$y'' + 2y' + y = u_1(t)e^{-(t-1)}\cos(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

6. (13 pts) Solve the following initial value problem:

$$y'' + y' - 6y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1.$$

7. (12 pts) Express the solution of the given initial value problem in terms of a convolution integral.

$$y'' + y' - 6y = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

   Notice that the left-hand side of the differential equation is identical to that of Problem 6.

Solutions to Common Exam III MATH 222 Nov. 18 2015

Prob 1: (a) $y'' + xy' + 2y = 0, \quad y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$y'' + xy' + 2y = 0, \quad \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$(n+2)(n+1) a_{n+2} + n a_n + 2a_n = 0$

$a_{n+2} = -\frac{(n+2)}{(n+2)(n+1)} a_n = -\frac{1}{n+1} a_n$

(b) $a_2 = -\frac{1}{1} \cdot a_0, \quad a_4 = -\frac{1}{3} a_2 = -\frac{1}{3}\left(-\frac{1}{1}\right) a_0 = \frac{1}{3} a_0$

$a_3 = -\frac{1}{2} a_1, \quad a_5 = -\frac{1}{4} a_3 = -\frac{1}{4}\left(-\frac{1}{2} a_1\right) = \frac{1}{8} a_1$

$y_1 = a_0 \left(1 - x^2 + \frac{1}{3} x^4 + \cdots\right)$

$y_2 = a_1 \left(x - \frac{1}{2} x^3 + \frac{1}{8} x^5 - \cdots\right)$

Prob 2: $x^2 y'' - 3xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 0$

$y = x^r$,

$r(r-1) - 3r + 4 = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0, \quad r = 2, 2$

$y_1 = x^2$

$y_2 = \ln x \cdot x^2$

$y = c_1 x^2 + c_2 x^2 \ln x$

$y' = 2c_1 x + 2c_2 x \ln x + c_2 x$

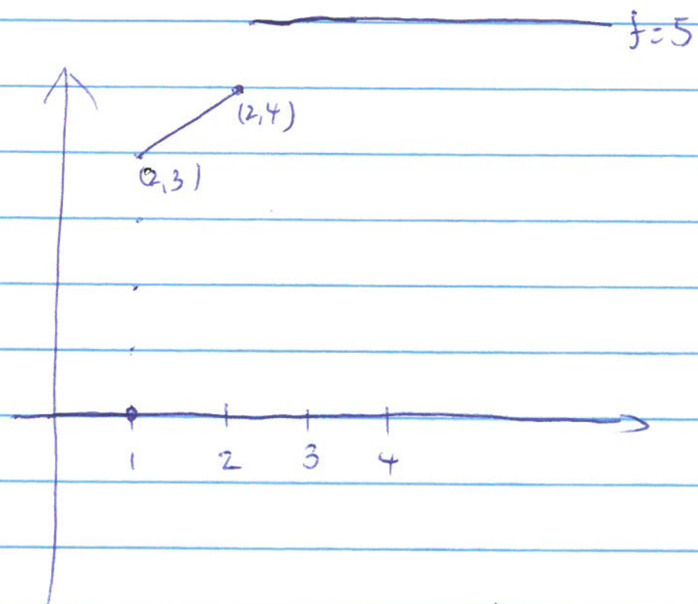
$y(1) = c_1 = 1$

$y'(1) = 2c_1 + c_2 = 0, \quad c_2 = -2, \quad \boxed{y(x) = x^2 - 2x^2 \ln x}$

P. 2

Prob. 3

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 2+t & 1 \le t < 2 \\ 5 & t \ge 2 \end{cases}$$



$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} \cdot 0 \, dt + \int_1^2 e^{-st}(2+t)\,dt + \int_2^\infty 5e^{-st}\,dt$$

$$= 2\int_1^2 e^{-st}\,dt + \int_1^2 t e^{-st}\,dt + 5\int_2^\infty e^{-st}\,dt$$

$$= \frac{2}{-s} e^{-st}\Big|_1^2 + \frac{5}{-s}\Big\{ e^{-st}\Big|_2^\infty + \int_1^2 t e^{-st}\,dt$$

$$= \frac{2}{-s} e^{-2s} + \frac{2}{s} e^{-s} + \frac{5}{-s}\left(\cancel{e^{-s\cdot\infty}} - e^{-2s}\right)$$

$$+ \int_1^2 t e^{-st}\,dt$$

$$= \frac{3}{s} e^{-2s} + \frac{2}{s} e^{-s} + \int_1^2 t e^{-st}\,dt$$

P. 3

$$\int_1^2 te^{-st}dt = t \cdot \frac{e^{-st}}{-s}\Big|_1^2 - \int_1^2 \frac{e^{-st}}{-s}dt = \frac{2e^{-2s}}{-s} - \frac{e^{-s}}{-s} + \frac{1}{s}\int_1^2 e^{-st}dt$$

$$= -\frac{2}{s}e^{-2s} + \frac{e^{-s}}{s} + \frac{1}{s}\cdot\left(-\frac{1}{s}e^{-st}\right)\Big|_1^2$$

$$= -\frac{2}{s}e^{-2s} + \frac{e^{-s}}{s} - \frac{1}{s^2}\left(e^{-2s} - e^{-s}\right)$$

$$\therefore \mathcal{L}[f(t)] = \frac{3}{s}e^{-2s} + \frac{2}{s}e^{-s} - \frac{2}{s}e^{-2s} + \frac{e^{-s}}{s} - \frac{1}{s^2}(e^{-2s} - e^{-s})$$

$$= \frac{1}{s}e^{-2s} + \frac{3}{s}e^{-s} - \frac{1}{s^2}(e^{-2s} - e^{-s})$$

Prob 4. (a) $\mathcal{L}^{-1}\left[\frac{e^{-3s}(s-2)}{s^2+4s+8}\right] = \mathcal{L}^{-1}\left[\frac{(s+2-4)e^{-3s}}{(s+2)^2+2^2}\right]$

$$\mathcal{L}^{-1}\left[\frac{s+2-4}{(s+2)^2+2^2}\right] = e^{-2t}\cos 2t - 2\cdot e^{-2t}\sin 2t$$

$$\mathcal{L}^{-1}\left[\frac{e^{-3s}(s-2)}{s^2+4s+8}\right] = u_3(t)f(t-3)$$

$$f(t) = e^{-2t}\cos 2t - 2e^{-2t}\sin 2t$$

(b) $\mathcal{L}^{-1}\left[\frac{s^2-4s+6}{(s-2)^3}\right] = \mathcal{L}^{-1}\left[\frac{(s-2)^2+2}{(s-2)^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-2} + \frac{2}{(s-2)^3}\right]$

$$= e^{2t} + t^2 e^{2t}$$

from table #2 & #19 with n=2

P.4

Prob 5 $y'' + 2y' + y = u_1 e^{-(t-1)} \cos(t-1)$ $y(0)=0,\ y'(0)=0$

$$(s^2+2s+1)Y = e^{-s} \cdot \mathcal{L}[e^{-t}\cos t]$$
$$= e^{-s} \cdot \frac{s+1}{(s+1)^2+1^2}$$

$$Y = \frac{s+1}{(s+1)^2+1} \cdot \frac{1}{(s+1)^2} e^{-s}$$

$$= \frac{1}{(s+1)^2+1} \cdot \frac{1}{s+1} \cdot e^{-s}$$

$$= \left(\frac{As+B}{(s+1)^2+1} + \frac{C}{s+1} \right) e^{-s}$$

$$(As+B)[s+1] + C[(s+1)^2+1] = 1$$

s^2: $A + C = 0$

s^1: $A+B + 2C = 0$ $B+C=0,\ B=-C$

s^0: $B + 2C = 1$ $C=1,\ B=-1,\ A=-1$

$$Y = \left(\frac{-s-1}{(s+1)^2+1} + \frac{1}{s+1} \right) e^{-s}$$

$$y(t) = -u_1 e^{-(t-1)} \cos(t-1) + e^{-(t-1)} u_1$$

P. 5

Prob 6 $y'' + y' - 6y = \delta(t-2\pi), \quad y(0)=0, \quad y'(0)=1$

$$s^2 Y - sy(0) - y'(0) + sY - y(0) - 6Y = e^{-2\pi s}$$

$$(s^2+s-6)Y - 1 = e^{-2\pi s}$$

$$(s^2+s-6)Y = 1 + e^{-2\pi s}, \quad Y = \frac{1+e^{-2\pi s}}{s^2+s-6}$$

$$Y = \frac{1}{(s+3)(s-2)}\left(1+e^{-2\pi s}\right)$$

$$= \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right)\left(1+e^{-2\pi s}\right)$$

$$\mathcal{L}^{-1}[Y] = \frac{1}{5}\left(e^{2t} - e^{-3t}\right) + u_{2\pi}\cdot\frac{1}{5}\left(e^{2(t-2\pi)} - e^{-3(t-2\pi)}\right)$$

Prob 7: $(s^2+s-6)Y = G$

$$Y = \frac{G}{s^2+s-6}$$

$$\mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{G}{s^2+s-6}\right] = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{G}{s-2} - \frac{G}{s+3}\right]$$

$$= \frac{1}{5}\int_0^t g(t-\tau)\left(e^{2\tau} - e^{-3\tau}\right)d\tau$$

or

$$\mathcal{L}^{-1}[Y] = \frac{1}{5}\int_0^t g(\tau)\left(e^{2(t-\tau)} - e^{-3(t-\tau)}\right)d\tau$$