1a) $ty' + y = e^t \quad y(1) = 0$

$y' + \frac{1}{t}y = \frac{e^t}{t}$

$\mu = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

$(\mu y)' = \mu \frac{1}{t} e^t$

$(ty)' = e^t$

$ty = e^t + C$

$y = \frac{e^t + C}{t}$

$0 = e + C$

$C = -e$

$$y = \frac{e^t - e}{t}$$

2b) $e^x y' - e^y = 0$ $\quad y(0) = 0$

$$e^x \frac{dy}{dx} = e^y$$

$$\frac{dy}{e^y} = \frac{dx}{e^x}$$

$$\int e^{-y}\,dy = \int e^{-x}\,dx$$

$$-e^{-y} = -e^{-x} + C$$

$$e^{-y} = e^{-x} + D$$

$$1 = 1 + D$$

$$D = 0$$

$$e^{-y} = e^{-x}$$

$$\boxed{y = x}$$

2a) $2y'' - y' - y = 0$ $\quad y(0) = 0$, $y'(0) = 1$

$2r^2 - r - 1 = 0$

$(2r + 1)(r - 1) = 0$

$r_1 = \frac{-1}{2} \quad r_2 = 1$

$y = c_1 e^{\frac{-t}{2}} + c_2 e^{t}$

$y' = \frac{-1}{2} c_1 e^{\frac{-t}{2}} + c_2 e^{t}$

$0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$1 = \frac{-1}{2} c_1 + c_2 \Rightarrow 1 = \frac{1}{2} c_2 + c_2$

$\frac{3}{2} c_2 = 1$

$c_2 = \frac{2}{3}$

$c_1 = \frac{-2}{3}$

$$y = \frac{-2}{3} e^{\frac{-t}{2}} + \frac{2}{3} e^{t}$$

2b)

$y'' = 1$ $\qquad$ $y'(0) = 0$

$y'(1) = 1$

$y' = t + C$

$y = \frac{1}{2}t^2 + Ct + D$

$0 = 0 + C$

$C = 0$

$1 = 1 + C$

$C = 0$

} Check Both Boundary Conditions

Values for C match

No info on D

So set $D = 1$

$$\boxed{y = \frac{1}{2}t^2 + 1}$$

3a)

$$y'' + y' - 2y = 6e^t$$

$$y'' + y' - 2y = 0$$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r = -2$$

$$r = 1$$

$$y_1 = e^{-2t}$$

$$y_2 = e^t$$

$$y_p = Ate^t$$

$$y_p' = Ae^t + Ate^t$$

$$y_p'' = Ae^t + Ae^t + Ate^t$$

$$y_p'' = 2Ae^t + Ate^t$$

$$2Ae^t + Ate^t + Ae^t + Ate^t - 2Ate^t = 6e^t$$

$$3Ae^t = 6e^t$$

$$3A = 6 \Rightarrow A = 2$$

$$\boxed{y_p = 2te^t}$$

3b) $y'' - y' = \delta(t-1)$ $\quad y(0)=0,\ y'(0)=0$

$\mathcal{L}[y''] - \mathcal{L}[y'] = \mathcal{L}[\delta(t-1)]$

$s^2\mathcal{L}[y] - sy(0) - y'(0) - s\mathcal{L}[y] + y(0) = e^{-s}$

$(s^2-s)\mathcal{L}[y] = e^{-s}$

$\mathcal{L}[y] = \frac{1}{s(s-1)} e^{-s}$

$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$

$1 = A(s-1) + Bs$

$A + B = 0 \Longrightarrow \boxed{B = 1}$

$-A = 1 \Rightarrow \boxed{A = -1}$

$\mathcal{L}[y] = e^{-s}\left[\frac{1}{s-1} - \frac{1}{s}\right]$

$F(s) = \frac{1}{s-1} - \frac{1}{s}$

$f(t) = e^t - 1$

$f(t-1) = e^{t-1} - 1$

$$y = u_1(t)\, f(t-1)$$

$$y = u_1(t)\left(e^{t-1} - 1\right)$$

4a)

$$f(t) = u_1(t)e^t = \begin{cases} 0 & 0 \le t < 1 \\ e^t & 1 \le t \end{cases}$$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^1 0\,dt + \int_1^\infty e^t e^{-st}dt$$

$$= \int_1^\infty e^{(1-s)t}dt$$

$$= \frac{1}{1-s}e^{(1-s)t}\Big|_1^\infty$$

$$= \frac{1}{1-s}e^{(1-s)\infty} + \frac{1}{s-1}e^{1-s} \quad ; s > 1$$

$$= \boxed{\frac{1}{s-1}e^{1-s}}$$

To check, use table

$$\mathcal{L}[u_1(t)e^t]$$

$$e\,\mathcal{L}[u_1(t)e^{t-1}]$$

$$e\,\frac{1}{s-1}e^{-s} = \frac{1}{s-1}e^{1-s}$$

46) $$F(s) = \frac{e^{-4s}}{4s+1}$$

$$F(s) = \frac{1}{4} e^{-4s} \frac{1}{s+1/4}$$

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{4}\mathcal{L}^{-1}\left[e^{-4s} \frac{1}{s+\frac{1}{4}}\right]$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[e^{-4s} \frac{1/2}{s+1/4}\right]$$

$$G(s) = \frac{1/2}{s+1/4}$$

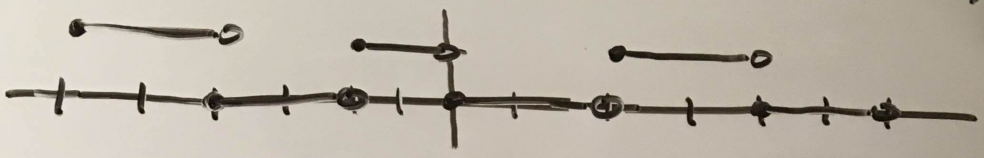
$$g(t) = \sin\left(\frac{t}{2}\right)$$

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2} u_4(t)\, g(t-4)$$

$$f(t) = \frac{1}{2} u_4(t)\, g(t-4)$$

$$f(t) = \frac{1}{2} u_4(t) \sin\left(\frac{t-4}{2}\right)$$

4 c) $f(x) = \begin{cases} 1 & -2 \le x < 0 \\ 0 & 0 \le x < 2 \end{cases}$ $f(x+4) = f(x)$



$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$T = 2L = 4$

$\boxed{L = 2}$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 1\,dx + \frac{1}{2}\int_{0}^{2} 0\,dx$$

$$a_0 = \frac{1}{2}x\Big|_{-2}^{0} = 1$$

$$\boxed{\frac{a_0}{2} = \frac{1}{2}}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} \cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} 0\,dx$$

$$= \frac{1}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0}$$

$$= \frac{1}{n\pi}\sin(n\pi)$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} \sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} 0\,dx$$

$$= \frac{-1}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{0}$$

$$= \frac{-1}{n\pi} + \frac{1}{n\pi}\cos(n\pi)$$

$$\boxed{b_n = \frac{-1}{n\pi} + \frac{1}{n\pi}(-1)^n}$$

$\sin(n\pi) = 0$

$\cos(n\pi) = (-1)^n$

5 a)

$$\vec{x}' = \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix} \vec{x} \quad : \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda \vec{x} = \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & -1/2 \\ 1/2 & 1-\lambda \end{vmatrix} \Rightarrow (\lambda-1)(\lambda-1) + \frac{1}{4} = 0$$

$$\lambda^2 - 2\lambda + 1 + \frac{1}{4} = 0$$

$$\lambda^2 - 2\lambda + \frac{5}{4} = 0$$

$$4\lambda^2 - 8\lambda + 5 = 0$$

$$\frac{8 \pm \sqrt{64 - 4(4)(5)}}{8} = \frac{8 \pm \sqrt{64-80}}{8}$$

$$= \frac{8 \pm \sqrt{-16}}{8} = \frac{8 \pm 4i}{8} = 1 \pm \frac{1}{2}i$$

$\lambda = 1 + \frac{1}{2}i$

$$\begin{bmatrix} 1-(1+1/2\,i) & -1/2 \\ 1/2 & 1-(1+1/2\,i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1/2\,i & -1/2 \\ 1/2 & -1/2\,i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{-1}{2}i\,x_1 - \frac{1}{2}x_2 = 0$$

$$i\,x_1 = -x_2$$

$$\frac{x_1}{x_2} = \frac{-1}{i} = \frac{i}{1}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 1 - \frac{1}{2}i$

$$\begin{bmatrix} 1-(1-1/2\,i) & -1/2 \\ 1/2 & 1-(1-1/2\,i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/2\,i & -1/2 \\ 1/2 & 1/2\,i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i\,\tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 = 0$$

$$\frac{x_1}{x_2} = \frac{1}{i} = \frac{-i}{1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\vec{x} = e^t\left[c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos\left(\tfrac{1}{2}t\right) + c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin\left(\tfrac{1}{2}t\right)\right]$$
$$+ e^t\left[c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos\left(\tfrac{1}{2}t\right) + c_2\begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin\left(\tfrac{1}{2}t\right)\right]$$

5b) $y'' + \lambda y = 0$ $\quad y'(0) = 0 \quad y(\pi) = 0$

$\lambda = 0$

$y'' = 0$

$y = Ax + B$

$y' = A$

$0 = A\pi + B$

$A = 0$

$B = 0$

Trivial sol.

$\lambda = 0$ is not an eisen value

$\lambda > 0$

$y'' + \lambda y = 0$

$r^2 + \lambda = 0$

$r^2 = -\lambda$

$r = \pm\sqrt{-\lambda}$

$r = \pm i\sqrt{\lambda}$

$y = c_1 \cos(x\sqrt{\lambda}) + c_2 \sin(x\sqrt{\lambda})$

$y' = -\sqrt{\lambda}\, c_1 \sin(x\sqrt{\lambda}) + \sqrt{\lambda}\, c_2 \cos(x\sqrt{\lambda})$

$c_1 = 0$

$0 = \sqrt{\lambda}\, c_2 \cos(\pi\sqrt{\lambda})$

$\cos(\pi\sqrt{\lambda}) = 0$

$\pi\sqrt{\lambda} = \frac{\pi}{2} + n\pi$

$\sqrt{\lambda} = \frac{1}{2} + n$

$\lambda = \left(\frac{1}{2} + n\right)^2$: eisen values

$y = \sin\left[\left(\frac{1}{2} + n\right)x\right]$: eisen function

$\lambda < 0$

$y'' + \lambda y = 0$

$r^2 = -\lambda$

$r = \pm\sqrt{-\lambda}$

$y = c_1 e^{x\sqrt{-\lambda}} + c_2 e^{-x\sqrt{-\lambda}}$

$y' = \sqrt{-\lambda}\, c_1 e^{x\sqrt{-\lambda}} - \sqrt{-\lambda}\, c_2 e^{-x\sqrt{-\lambda}}$

$0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$0 = \sqrt{-\lambda}\, c_1 e^{\pi\sqrt{-\lambda}} - \sqrt{-\lambda}\, c_2 e^{-\pi\sqrt{-\lambda}}$

$c_1 e^{\pi\sqrt{-\lambda}} = -c_1 e^{-\pi\sqrt{-\lambda}} \Rightarrow c_1 = c_2 = 0$

$\Rightarrow \lambda < 0$ is not an eisen value

6a) $f(x) = xe^{x} - xe^{-x}$

$f(-x) = -xe^{-x} + xe^{x} = xe^{x} - xe^{-x} = f(x)$

$f(-x) = f(x)$

$\therefore f(x)$ is even

6 b) i) $g(t) = t\,u_1(t) - u_3(t)(t-3)$

$g(2) = 2u_1(2) - u_3(2)(2-3)$

$\boxed{g(2) = 2}$

ii) $\lim_{t\to\infty} t\,u_1(t) - u_3(t)(t-3)$

$\lim_{t\to\infty} t - (t-3)$

$\lim_{t\to\infty} 3 = \boxed{3}$

6c) $f(x) = \cos x, \quad 0 < x < \frac{\pi}{2}$

