Math 222 Final Exam December 14, 2007

1 a) $x' = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - (-1)(2) \Rightarrow \lambda^2 - 4\lambda + 3 + 2$$

$$\lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16-20}}{2}$$

$(\lambda-2)^2 + 1 = 0$

$\lambda - 2 = \pm i$

$\lambda = 2 \pm i$

$$\lambda = 2 \pm \frac{\sqrt{-4}}{2} \Rightarrow \lambda = 2 \pm i$$

$\lambda = 2+i \qquad \begin{pmatrix} 1-2-i & -1 \\ 2 & 3-2-i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$(-1-i)x_1 - x_2 = 0 \qquad \frac{x_1}{x_2} = \frac{1}{-1-i} = \frac{-1}{1+i} = \frac{-(1-i)}{2}$$

$$\vec{\lambda} = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda = 2-i$

$$\begin{pmatrix} 1-2+i & -1 \\ 2 & 3-2+i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$(-1+i)\,x_1 - x_2 = 0 \Rightarrow \frac{x_1}{x_2} = \frac{1}{-1+i} \cdot \frac{(1+i)}{(1+i)}$$

$$\frac{x_1}{x_2} = \frac{1+i}{i^2-1} = \frac{1+i}{-2}$$

$$\vec{\lambda}_2 = \begin{pmatrix} 1+i \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x(t) = c_1 e^{2t}\left(\cos(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$+ c_2 e^{2t}\left(\sin(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -2c_1 \end{pmatrix}$$

$$c_1 = \frac{1}{-2} \qquad c_1 + c_2 = 1$$
$$c_2 = \frac{3}{2}$$

$$\boxed{x(t) = \frac{-1}{2} e^{2t}\left(\cos(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \frac{3}{2} e^{2t}\left(\sin(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}$$

1 b) $y'' + \lambda y = 0 \qquad y'(0) = 0 \;\&\; y'(L) = 0$

$$y'' + \lambda y \Rightarrow r^2 + \lambda = 0$$

$$r^2 = -\lambda$$

$$r = \sqrt{-\lambda}$$

$\lambda = 0 \qquad y'' = 0$

$$\left.\begin{array}{ll} y = Ax + B & \\ y' = A & y'(0) = A = 0 \\ & y'(L) = A = 0 \end{array}\right\}$$ Non trivial Solution $\therefore$ $\lambda = 0$ could be a solution.

$\lambda < 0 \qquad r^2 = \sqrt{-\lambda} \Rightarrow r = \pm\sqrt{-\lambda}$

$$y = C_1 e^{\sqrt{-\lambda}\,x} + C_2 e^{-\sqrt{-\lambda}\,x}$$

$$y' = \sqrt{-\lambda}\, C_1 e^{\sqrt{-\lambda}\,x} + (-\sqrt{-\lambda})\, C_2 e^{-\sqrt{-\lambda}\,x}$$

$$\left.\begin{array}{l} y'(0) = \sqrt{-\lambda}\, C_1 - \sqrt{-\lambda}\, C_2 = 0 \\ y'(L) = \sqrt{-\lambda}\, C_1 e^{\sqrt{-\lambda}\,L} + (-\sqrt{-\lambda})\, C_2 e^{-(\sqrt{-\lambda})L} \end{array}\right\}$$ $C_1 = C_2 = 0$ $\therefore$ Trivial Solution.

$\lambda > 0$ $\qquad r^2 = \sqrt{-\lambda}$

$r = \pm\sqrt{\lambda}\, i$

$y = D_1 \sin(\sqrt{\lambda}\, x) + D_2 \cos(\sqrt{\lambda}\, x)$

$y' = \sqrt{\lambda}\, D_1 \cos(\sqrt{\lambda}\, x) - \sqrt{\lambda}\, D_2 \sin(\sqrt{\lambda}\, x)$

$y'(0) = \sqrt{\lambda}\, D_1 = 0 \qquad \Rightarrow D_1 = 0$

$y'(L) = -\sqrt{\lambda}\, D_2 \sin(\sqrt{\lambda}\, L) = 0$

D_2 cannot be zero, So,

$$\sin(\sqrt{\lambda}\, L) = \sin(n\pi) = 0$$

$$\sqrt{\lambda}\, L = n\pi$$

$$\sqrt{\lambda} = \frac{n\pi}{L}$$

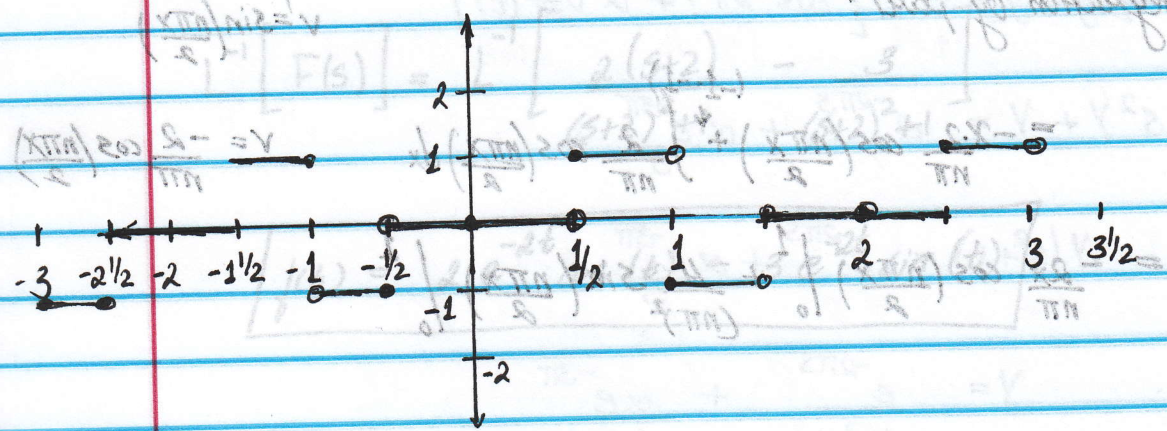
$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$\boxed{y = D_2 \cos(\sqrt{\lambda}\, x) \quad \text{where } \lambda = \left(\frac{n\pi}{L}\right)^2}$$

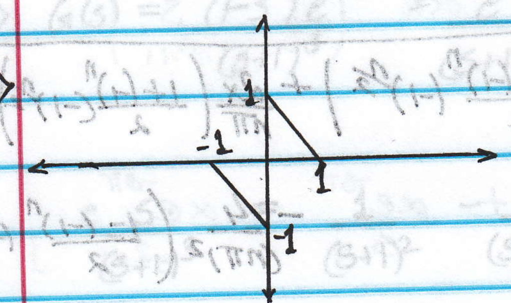
2 a) $L=2$

$$f(x)=\begin{cases}0 & 0\le x<\frac{1}{2}\\ 1 & \frac{1}{2}\le x<1\end{cases}$$

Sketch over $[-3,3]$



2 b)



$f(x)=1-x \qquad 0\le x<1$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n=\frac{1}{2}\int_{-1}^{1}(1-x)\sin\left(\frac{n\pi x}{2}\right)dx \Rightarrow b_n=\int_0^1(1-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$b_n=\int_0^1 \sin\left(\frac{n\pi x}{2}\right)dx-\underbrace{\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx}_{\text{Integration by Part.}}$$

$$b_n = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1$$

$u = x \quad du = dx$

$v' = \sin\left(\frac{n\pi x}{2}\right)$

$v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$

Integration by part:

$$= -x\cdot\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) \overset{(-1\cdot -1)}{+} \int \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1$$

$$b_n = -\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) + \frac{2x}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) - \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\boxed{b_n = \cancel{\frac{2}{n\pi}} - \cancel{\frac{2x}{n\pi}} - \frac{2}{n\pi}\left(\frac{1+(-1)^n}{2}(-1)^{\frac{n}{2}}\right) + \frac{2x}{n\pi}\left(\frac{1+(-1)^n}{2}(-1)^{\frac{n}{2}}\right) - \frac{4}{(n\pi)^2}\left(\frac{1-(-1)^n}{2}(-1)^{\frac{n-1}{2}}\right)}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

3 a) $F(s) = \dfrac{2s+1}{s^2+2s+5} = \dfrac{2s+1}{s^2+2s+4+1} = \dfrac{2s+1}{(s+2)^2+1} = \dfrac{2s+4}{(s+2)^2+1} - \dfrac{3}{(s+2)^2+1}$

$$L^{-1}\left[F(s)\right] = L^{-1}\left[\frac{2(s+2)}{(s+2)^2+1} - \frac{3}{(s+2)^2+1}\right]$$

$$\boxed{f(t) = 2e^{-2t}\cos(t) - 3e^{-2t}\sin(t)}$$

3 b) $G(s) = \dfrac{(1-s)e^{-2s}}{(s+1)^3} = \dfrac{e^{-2s}}{(s+1)^3} - \dfrac{se^{-2s}}{(s+1)^3}$ } Partial Fraction.

$$\frac{s}{(s+1)^3} = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3} \rightarrow L^{-1} = e^{-2s}\left(\frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}\right)$$

$$L^{-1}\left[G(s)\right] = L^{-1}\left[\frac{e^{-2s}}{(s+1)^3} - \frac{e^{-2s}}{(s+1)^2} + \frac{e^{-2s}}{(s+1)^3}\right]$$

$$= L^{-1}\left[\frac{2e^{-2s}}{(s+1)^3} - \frac{e^{-2s}}{(s+1)^2}\right] =$$

$$\boxed{\begin{array}{ll} g(t) = u_2(t)\, f(t-2) & -\; u_2(t)\, h(t-2) \\ \quad f(t) = e^{-t}t^2 & \quad h(t) = e^{-t}\cdot t \end{array}}$$

4 a) $y'' + y = \delta(t-2\pi) + \alpha u_{3\pi}(t)$ $\qquad y(0)=0 \quad y'(0)=0$

$$L[y''+y] = L[\delta(t-2\pi) + \alpha u_{3\pi}(t)]$$

$$s^2 Y + Y = e^{-2\pi s} + \alpha \frac{e^{-3\pi}}{s}$$

$$Y(s^2+1) = e^{-2\pi s} + \alpha \frac{e^{-3\pi}}{s}$$

$$Y = \frac{e^{-2\pi s}}{(s^2+1)} + \frac{\alpha e^{-3\pi}}{s(s^2+1)}$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \qquad \alpha e^{-3\pi}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$L^{-1}[Y] = L^{-1}\left[\frac{e^{-2\pi s}}{s^2+1} + \frac{\alpha e^{-3\pi}}{s} - \frac{\alpha e^{-3\pi} s}{s^2+1}\right]$$

$$\boxed{y(t) = u_{2\pi}(t) f(t-2\pi) + \alpha u_{3\pi}(t) - \alpha u_{3\pi}(t) g(t-3\pi) \qquad f(t) = \sin(t) \qquad g(t) = \cos(t)}$$

4 b) $y(t) = u_{2\pi}(t) f(t-2\pi) + \alpha\, u_{3\pi}(t) - \alpha\, u_{3\pi}(t)\, g(t-3\pi)$

$f(t) = \sin(t)$ $\qquad$ $g(t) = \cos(t)$

$y\left(\frac{5\pi}{2}\right) = u_{2\pi}\left(\frac{5\pi}{2}\right) \sin(t-2\pi) + \alpha\, u_{3\pi}\left(\frac{5\pi}{2}\right) - \alpha\, u_{3\pi}\left(\frac{5\pi}{2}\right)\cos(t-3\pi)$

$= \sin\left(\frac{5\pi}{2} - 2\pi\right) + \alpha \cdot \underbrace{\cancel{u_{3\pi}\left(\frac{5\pi}{2}\right)}}_{0 \text{ because } t<c} - \alpha\, \underbrace{\cancel{u_{3\pi}\left(\frac{5\pi}{2}\right)}}_{0 \text{ because } t<c} \cos\left(\frac{5\pi}{2} - 3\pi\right)$

$c = 3\pi$ $\qquad$ $c = 3\pi$

$$\boxed{y\left(\frac{5\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1}$$

4 c) $y\left(\frac{7\pi}{2}\right) = 0$

$y\left(\frac{7\pi}{2}\right) = u_{2\pi}\left(\frac{7\pi}{2}\right) \sin\left(\frac{7\pi}{2} - 2\pi\right) + \alpha\, u_{3\pi}\left(\frac{7\pi}{2}\right) - \alpha\, u_{3\pi}\left(\frac{7\pi}{2}\right) \cos\left(\frac{7\pi}{2} - 3\pi\right)$

$y\left(\frac{7\pi}{2}\right) = \sin\left(\frac{7\pi}{2} - 2\pi\right) + \alpha - \alpha \cancel{\cos\left(\frac{7\pi}{2} - 3\pi\right)}^{\,0} = 0$

$y\left(\frac{7\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) + \alpha = 0$

$\alpha = -\sin\left(\frac{3\pi}{2}\right)$

$$\boxed{\alpha = -(-1) = 1}$$

5a) $ty' + 2y = \frac{\cos(t)}{t}$ $\quad t>0 \quad y(\pi)=0$

$\boxed{\text{First Order Linear}}$

$$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}$$

$$\mu = e^{\int q\,dt} = e^{\int 2/t\,dt} = e^{2\ln(t)} = t^2$$

$$\mu = t^2$$

$$(\mu y)' = \mu g(t)$$

$$y = \frac{\int \cancel{t^2}\frac{\cos(t)}{\cancel{t^2}}}{t^2}$$

$$y = \frac{\int \cos(t)}{t^2} \Rightarrow y = \frac{\sin(t) + C}{t^2}$$

$$y(\pi) = 0 = \frac{\cancelto{0}{\sin(\pi)} + C}{\pi^2} \Rightarrow C = 0$$

$$\boxed{y = \frac{\sin(t)}{t^2}}$$

5 b) $y' = \dfrac{2x}{y+x^2y}$ $\quad y(0) = -2$

First Order Non Linear.

$$y'(y) = \frac{2x}{(x^2+1)} \Rightarrow \frac{dy}{dx}(y) = \frac{2x}{(x^2+1)}$$

$$y\,dy = \frac{2x}{x^2+1}\,dx \Rightarrow \int y\,dy = \int \frac{2x}{x^2+1}\,dx$$

$$\frac{y^2}{2} = (\ln(x^2+1)) + C$$

$$y^2 = 2\ln(x^2+1) + 2C$$

$$y = \sqrt{\ln(x^2+1)^2 + \sqrt{2C}}$$

$$y(0) = -2 = \sqrt{2C} \Rightarrow 4 = 2C$$

$$C = 2$$

$$y = \sqrt{\ln(x^2+1)^2 + 4}$$

6 a) $y'' + 6y' + 13y = e^{-3x}\sin(2x) + x^2\cos(3x)$

$y'' + 6y' + 13y = 0$

$r^2 + 6r + 13 = 0$

$r^2 + 6r + 9 + 4 = 0$

$r^2 + 6r + 9 = -4$

$(r+3)^2 = -4$

$r = -3 \pm 2i \Rightarrow \alpha = 3 \quad \beta = 2$

$y = e^{-3x}\left(C_1\cos(2x) + C_2\sin(2x)\right) + Y_p$

$$\boxed{Y_p = e^{-3x}\left(\sin(2x) + \cos(2x)\right)\cdot x + (Ax^2 + Bx + C)\left(\sin(3x) + \cos(3x)\right)}$$

6 b) $y_1 = x^2 \qquad y_2 = x^2 v(x) \qquad y_2' = 2x\,v(x) + v'(x)\,x^2$

$y_2'' = 2v(x) + 4x\,v'(x) + v''(x)\,x^2$

$x^2y'' - 3xy' + 4y = 0$

$x^2\left(2v(x) + 4x\,v'(x) + v''(x)\,x^2\right) - 3x\left(2x\,v(x) + v'(x)\,x^2\right)$

$+ 4\left(x^2 v(x)\right) = 0$

$\cancel{(2x^2 - 6x^2 + 4x^2)}^{0}\,v(x) + (4x^3 - 3x^3)\,v'(x) + x^4\,v''(x) = 0$

$v'(x)\cdot x^3 + v''(x)\,x^4 = 0$

$\dfrac{v''(x)}{v'(x)} = -\dfrac{x^3}{x^4} \Rightarrow \ln(v') = -\ln(x)$

$v' = x^{-1} \Rightarrow v = \ln(x)$

$$\boxed{y_2 = x^2\ln(x)}$$

$v' = x^{-1} \Rightarrow \int v' dx = \int x^{-1} dx$

$v = \ln(x)$

$\boxed{y_2 = x^2 \ln(x)}$