P.01

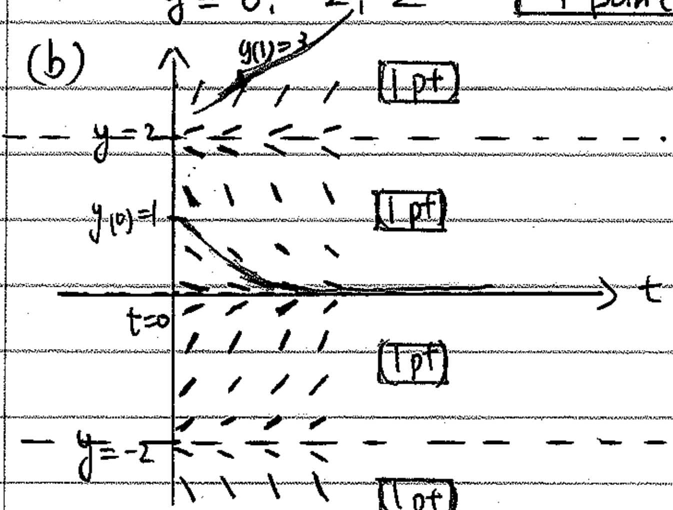
Grading Guidelines for Exam I, M222, Feb 17 2016

Problem 1: (a) $y' = y(y^2 - 4)$

$y' = 0 = y(y^2-4)$

$y = 0, -2, 2$ $\boxed{\text{1 point each}}$

(b)



$y > 2, \quad y' > 0$

$0 < y < 2, \quad y' < 0$

$-2 < y < 0, \quad y' > 0$

$y < -2, \quad y' < 0$

(c) see the red curves above

$\boxed{\text{2 pts for each curve}}$

Problem 2: (a) 2nd order, nonlinear differential equation

(2pts) (2pts)

(b) $y_1 = t^{1/2}, \quad y_1' = \frac{1}{2}t^{-1/2}, \quad y_1'' = -\frac{1}{4}t^{-3/2}$

$2t^2 y_1'' + 3t y_1' - y_1$

$= 2t^2\left(-\frac{1}{4} t^{-3/2}\right) + 3t\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2}$

$= -\frac{1}{2}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} = 0 \Rightarrow y_1$ is a solution $\boxed{\text{3pts}}$

$y_2 = t^{-1}, \quad y_2' = -t^{-2}, \quad y_2'' = 2t^{-3}$

$2t^2 y_2'' + 3t y_2' - y_2$

$= 2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0$

Grading Guidelines for Exam I, M222, Feb 17 2016

y_2 is a solution. $\boxed{3\text{pts}}$

Problem 3: (a) $y'' - \frac{3t}{t(t-4)}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$, $y(3)=0$, $y'(3)=-1$

intervals of t for twice-differentiable solutions $\Rightarrow$

$t \in (-\infty, 0)$ $\boxed{1\text{pt}}$

$t \in (0, 4)$ $\boxed{1\text{pt}}$

$t \in (4, \infty)$ $\boxed{1\text{pt}}$

as the initial values are assigned at $t=3$ $\Rightarrow$ the longest interval is $(0,4)$ $\boxed{2\text{pts}}$

(b) $W = c \cdot e^{\int -p\,dt}$, where $p(t) = -\frac{3}{t-4}$ $\boxed{2\text{pts}}$

$$\int \frac{3}{t-4}\,dt = 3\ln(t-4)$$

$$W = c \cdot e^{3\ln(t-4)} = c \cdot e^{\ln(t-4)^3} = c\cdot(t-4)^3 \quad \boxed{3\text{pts}}$$

Problem 4: (a) $\frac{dy}{dx} = \frac{x}{y(1+x^2)}$, $y(0) = -2$

$$y\,dy = \frac{x}{1+x^2}\,dx \quad \boxed{2\text{pts}}$$

$$\int y\,dy = \int \frac{x}{1+x^2}\,dx, \quad \frac{y^2}{2} = \frac{1}{2}\ln(1+x^2) + c$$

$$y^2 = \ln(1+x^2) + C \quad \boxed{2\text{pts}}$$

$x=0$, $4 = C$ $\Rightarrow$ $y = \pm\sqrt{\ln(1+x^2)+4}$

(1 pt)

$y = -\sqrt{\ln(1+x^2)+4}$ because $y(0) = -2$ $\boxed{2\text{pts}}$

Grading Guidelines for Exam I, M222, Feb17 2016

(b) $y' + y = e^{-t}$, $y(0) = y_0$

$\mu' = \mu$, $\mu = e^{t}$

$(e^{t}y)' = 1$, $e^{t}y = t + y_0$, $y = te^{-t} + y_0 e^{-t}$ $\boxed{5\text{pts}}$

$y' = -te^{-t} + e^{-t} - y_0 e^{-t} = 0$ at $t = 4$

$-4e^{-4} + e^{-4} - y_0 e^{-4} = 0$

$y_0 = -3$ $\boxed{3\text{pts}}$

Problem 5: (a) $\frac{dV}{dt} = 3 - 0.5V = 3 - \frac{1}{2}V$, $V(0) = 1$ (L) $\boxed{4\text{ pts}}$

$V' + \frac{1}{2}V = 3$, $\mu' = \frac{1}{2}\mu$, $\mu = e^{\frac{1}{2}t}$ $\boxed{2\text{pts}}$

$(e^{\frac{1}{2}t} \cdot V)' = 3e^{\frac{1}{2}t}$, $e^{\frac{1}{2}t} \cdot V = \int 3e^{\frac{1}{2}t}dt = 6e^{\frac{t}{2}} + C$

$V = 6 + C \cdot e^{-\frac{1}{2}t}$, $V(0) = 1$, $C = -5$

$\boxed{V(t) = 6 - 5e^{-\frac{1}{2}t}}$ $\boxed{4\text{pts}}$

(b) $\frac{dQ}{dt}$ = rate in − rate out

$Q' = 3 \cdot 1 - \frac{Q}{V} \frac{1}{2}V = 3 - \frac{1}{2}Q$, $Q(0) = 10\,g$ $\boxed{5\text{pts}}$

Problem 6: $y' = -y + 1 - t$, $y(0) = 1$, $h = 0.1$

$y_{n+1} = y_n + h \cdot (-y_n + 1 - t_n)$, $\boxed{3\text{ pts}}$

$n=1$, $t_1 = 0$, $y_1 = 1$

$n=2$, $t_2 = 0.1$, $y_2 = y_1 + 0.1 \cdot (-1 + 1 - 0.0) = y_1 = 1$ $\boxed{3\text{pts}}$

Grading Guidelines for Exam I, M222, Feb 17 2016

$n=3, \quad t_3 = 0.2, \quad y_3 = y_2 + 0.1 \cdot (-1 + 1 - 0.1)$

$$y_3 = 1 + 0.1(-0.1) = 1 - 0.01 = 0.99$$

$\boxed{4\text{pts}}$

Problem 7: (a) $(r+2)(r-3)=0, \quad r^2 - r - 6 = 0 \quad \boxed{2\text{pts}}$

$\Rightarrow y'' - y' - 6y = 0 \quad \boxed{3\text{pts}}$

(b) $y'' - y' - 2y = 0 \quad y(0) = \alpha, \quad y'(0) = \beta$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = -1, 2 \quad \boxed{2\text{pts}}$

$y(t) = c_1 e^{-t} + c_2 e^{2t} \quad \boxed{2\text{pts}}$

$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t}$

$y(0) = \alpha = c_1 + c_2$

$y'(0) = \beta = -c_1 + 2c_2$

$3c_2 = \alpha + \beta, \quad c_2 = \frac{\alpha+\beta}{3}$

$c_1 + \frac{\alpha+\beta}{3} = \alpha,$

$c_1 = \alpha - \frac{\alpha+\beta}{3} = \frac{2\alpha - \beta}{3}$

$$y(t) = \frac{2\alpha-\beta}{3} e^{-t} + \frac{\alpha+\beta}{3} e^{2t} \quad \boxed{4\text{ pts}}$$

want y to remain bounded as $t \to \infty$.

$\alpha + \beta = 0, \quad \boxed{\alpha = -\beta} \quad \boxed{2\text{pts}}$

(c) $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

$r^2 + 4r + 5 = 0 \quad (r+2)^2 + 1 = 0, \quad r = -2 \pm i \quad \boxed{3\text{pts}}$

$y = e^{-2t}(c_1 \cos t + c_2 \sin t) \quad \boxed{4\text{pts}}$

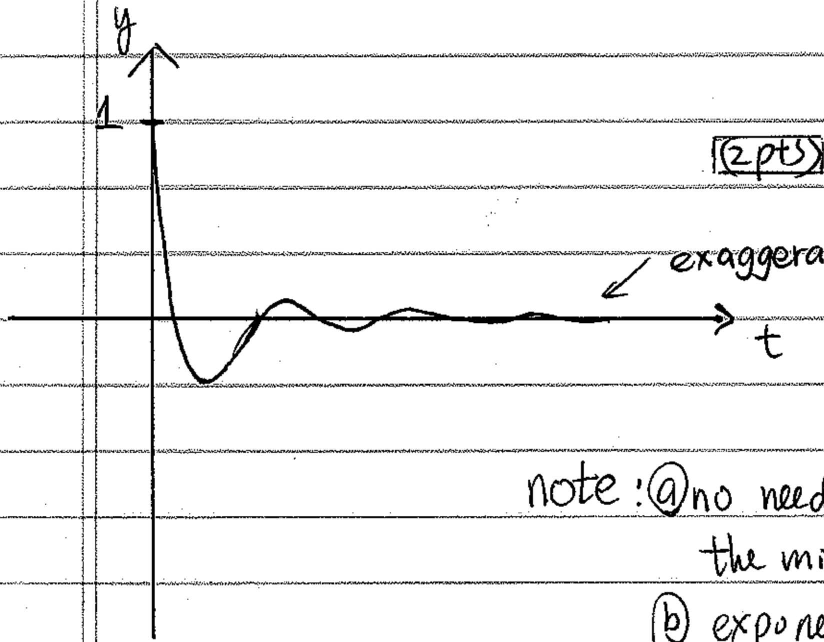
$y' = -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(-c_1 \sin t + c_2 \cos t)$

$y(0) = 1 = c_1 \quad \boxed{3\text{pts}}, \quad y'(0) = 0 = -2c_1 + c_2, \quad c_2 = 2 \quad \boxed{3\text{pts}}$

Grading Guidelines for Exam I, M222, Feb 17 2016

$$y(t) = e^{-2t}(\cos t - 2\sin t)$$

$$= \sqrt{5}e^{-2t}\left(\frac{1}{\sqrt{5}}\cos t - \frac{2}{\sqrt{5}}\sin t\right) = \sqrt{5}e^{-2t}\cdot\cos(t+\theta), \quad \cos\theta = \frac{1}{\sqrt{5}}, \quad \sin\theta = \frac{2}{\sqrt{5}}$$



$\boxed{(2\text{pts})}$

exaggerated, but as long as the general trend is given in the solution, the 2pts will be given.

note: ⓐ no need to provide the intersections, the minima/maxima, etc.

ⓑ exponential decaying trend → 1 pt

ⓒ oscillation → 1 pt