

P.01

Grading Guidelines for Exam II, M222, March 09 2016

Problem 1:

$$4r^2 + 12r + 9 = 0$$

$$(2r+3)^2 = 0, \quad r = -\frac{3}{2}, -\frac{3}{2} \quad \boxed{4 \text{ pts}}$$

$$y(t) = C_1 e^{-\frac{3}{2}t} + C_2 t e^{-\frac{3}{2}t} \quad \boxed{2 \text{ pts}}$$

$$y'(t) = -\frac{3}{2} C_1 e^{-\frac{3}{2}t} + C_2 e^{-\frac{3}{2}t} - \frac{3}{2} C_2 t e^{-\frac{3}{2}t}$$

$$y(0) = -1 = C_1 + 0, \quad C_1 = -1$$

$$y'(0) = \alpha = -\frac{3}{2} C_1 + C_2, \quad C_2 = \alpha + \frac{3}{2} C_1 = \alpha - \frac{3}{2}$$

$$y(t) = e^{-\frac{3}{2}t} \left(-1 + \left(\alpha - \frac{3}{2} \right) t \right) \quad \boxed{4 \text{ pts}}$$

$$y\left(\frac{1}{2}\right) = 0 = e^{-\frac{3}{4}} \cdot \left(-1 + \left(\alpha - \frac{3}{2} \right) \frac{1}{2} \right) \quad -1 + \left(\alpha - \frac{3}{2} \right) \frac{1}{2} = 0$$

$$\alpha - \frac{3}{2} = 2, \quad \boxed{\alpha = \frac{7}{2}} \quad \boxed{3 \text{ pts}}$$

$$y(t) = e^{-\frac{3}{2}t} (-1 + 2t) \Rightarrow \text{can only change signs once} \quad \boxed{2 \text{ pts}}$$

Problem 2: (a) $y_2 = v \cdot t^2, \quad y_2' = v' t^2 + 2vt, \quad y_2'' = v'' t^2 + 4v't + 2v$

$$t^2(t^2 v'' + 4t v' + 2v) - 3t(t^2 v' + 2t v) + 4t^2 v = 0$$

$$t^4 v'' + 4t^3 v' - 3t^3 v' = 0$$

$$t v'' + v' = 0 \quad \boxed{7 \text{ pts}}$$

$$\frac{v''}{v'} = -\frac{1}{t}, \quad \ln v' = -\ln t = \ln \frac{1}{t}, \quad v' = \frac{1}{t}, \quad v = \ln t$$

$$y_2 = \ln t \cdot t^2 \quad \boxed{3 \text{ pts}}$$

(b) $r^2 + 5r + 6 = 0, \quad (r+3)(r+2) = 0, \quad r = -3, -2$

$$Y = \underbrace{At+B}_{\boxed{1 \text{ pt}}} + \underbrace{t \cdot C e^{-3t}}_{\boxed{1 \text{ pt}}} + \underbrace{t(Dt+E)e^{-2t}}_{\boxed{2 \text{ pts}}} + \underbrace{e^{-3t}(F \cos t + G \sin t)}_{\boxed{1 \text{ pt}}}$$

P.02

Grading Guidelines for Exam II, M222, March 09, 2016

Problem 3:

$$r^2 + 2r + 2 = 0, \quad (r+1)^2 + 1 = 0, \quad r = -1 \pm i$$

$$y_1 = e^{-t} \cdot \cos t, \quad y_2 = e^{-t} \cdot \sin t \quad \boxed{4 \text{ pts}}$$

$$Y = A \cos t + B \sin t$$

$$Y' = -A \sin t + B \cos t$$

$$Y'' = -A \cos t - B \sin t$$

$$Y'' + 2Y' + 2Y = \cos t$$

$$-A \cos t - B \sin t + 2(-A \sin t + B \cos t) + 2(A \cos t + B \sin t) = \cos t$$

$$\begin{array}{lcl} \cos t: & -A + 2B + 2A = 1 & A + 2B = 1 \\ \sin t: & -B - 2A + 2B = 0 & -2A + B = 0 \end{array} \quad \begin{array}{l} 5A = 1 \\ B = 2A \end{array}$$

$$A = \frac{1}{5}, \quad B = \frac{2}{5}$$

$$Y = \frac{1}{5} \cos t + \frac{2}{5} \sin t \quad \boxed{6 \text{ pts}}$$

$$y = e^{-t} (C_1 \cos t + C_2 \sin t) + \frac{1}{5} \cos t + \frac{2}{5} \sin t$$

$$\text{as } t \rightarrow \infty \quad y \rightarrow \frac{1}{5} \cos t + \frac{2}{5} \sin t \quad \boxed{5 \text{ pts}}$$

Problem 4:

$$r^2 - 2r + 1 = 0, \quad r = 1, 1, \quad y_1 = e^t, \quad y_2 = te^t \quad \boxed{4 \text{ pts}}$$

$$Y = \left(\int \frac{-te^t \cdot e^t/t^2}{w(y_1, y_2)} dt \right) e^t + \left(\int \frac{e^t \cdot e^t/t^2}{w(y_1, y_2)} dt \right) te^t$$

$$\Rightarrow w(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$$

$$\begin{aligned} Y &= \left(\int -\frac{1}{t} dt \right) e^t + \left(\int \frac{1}{t^2} dt \right) te^t \\ &= -\ln t e^t - \frac{1}{t} \cdot te^t \quad \boxed{5 \text{ pts}} \end{aligned}$$

P.03

Grading Guidelines for Exam II, M222, March 09 2016

$$y = C_1 e^t + C_2 t e^t - \ln t e^t$$

$$y' = C_1 e^t + C_2 e^t + C_2 t e^t - \frac{e^t}{t} - \ln t e^t$$

$$y(1) = e = (C_1 + C_2)e$$

$$C_1 + C_2 = 1$$

$$C_1 = 1$$

$$y'(1) = 0 = (C_1 + C_2 + C_2 - 1)e$$

$$C_1 + 2C_2 = 1$$

$$C_2 = 0$$

$$y(t) = e^t - \ln t e^t = (1 - \ln t) e^t \quad \boxed{4 \text{ pts}}$$

$$\text{as } t \rightarrow \infty, y \rightarrow -\infty \quad \boxed{2 \text{ pts}}$$

Problem 5: (a) $k = \frac{0.5 \text{ ft}}{0.5 \text{ lb}} = 1 \text{ ft/lb}, m = \frac{1}{64} \text{ lb} \cdot \text{s}^2/\text{ft}$

$$\frac{1}{64} u'' + u = 0, \quad u(0) = \frac{1}{2}, \quad u'(0) = 0$$

$$u'' + 64u = 0, \quad \boxed{4 \text{ pts}}$$

$$u = A \cos 8t + B \sin 8t$$

$$u' = -8A \sin 8t + 8B \cos 8t$$

$$u(0) = A = \frac{1}{2}$$

$$u'(0) = 0 = B$$

$$u(t) = \frac{1}{2} \cos 8t$$

$$\text{return to equilibrium: } u = 0 = \frac{1}{2} \cos 8t, \quad \boxed{4 \text{ pts}}$$

$$8t = \frac{\pi}{2} + n\pi$$

$$t = \frac{\pi}{16} + \frac{8n\pi}{8}$$

$$\text{first return} \Rightarrow \boxed{t = \frac{\pi}{16} \text{ s}} \quad \boxed{4 \text{ pts}}$$

(b) $\gamma = 2, \quad \frac{1}{64} u'' + 2u' + u = 0, \quad u(0) = -\frac{1}{2}, \quad u'(0) = \frac{1}{2}$

$$u'' + 128u' + 64u = 0 \quad \boxed{6 \text{ pts}}$$

$$r^2 + 128r + 64 = 0, \quad r = \frac{-128 \pm \sqrt{128^2 - 4 \cdot 64}}{2}$$

P.04

Grading Guidelines for Exam II, M222, March 09 2016

$$r = -64 \pm \frac{1}{2} \sqrt{128^2 - 256} = -64 \pm \frac{1}{2} \sqrt{64 \cdot 2 \cdot 64 \cdot 2 - 64 \cdot 4}$$

$$= -64 \pm \frac{1}{2} \cdot \sqrt{64 \cdot 4} \cdot \sqrt{64 - 1}$$

$$u = c_1 e^{(-64+8\sqrt{63})t} + c_2 e^{(-64-8\sqrt{63})t}$$

$$u' = (-64+8\sqrt{63}) c_1 e^{(-64+8\sqrt{63})t} + (-64-8\sqrt{63}) c_2 e^{(-64-8\sqrt{63})t}$$

$$u(0) = -\frac{1}{2} = c_1 + c_2 \quad c_1 = -\frac{1}{2} - c_2 \quad \left. \begin{array}{l} \text{3 pts} \end{array} \right\}$$

$$u'(0) = \frac{1}{2} = (-64+8\sqrt{63}) c_1 + (-64-8\sqrt{63}) c_2 \quad \left. \begin{array}{l} \text{3 pts} \end{array} \right\}$$

$$(-64+8\sqrt{63}) \left(-\frac{1}{2} - c_2\right) + (-64-8\sqrt{63}) c_2 = \frac{1}{2}$$

$$-\frac{1}{2}(-64+8\sqrt{63}) - 16\sqrt{63} c_2 = \frac{1}{2}$$

$$32 - 4\sqrt{63} - \frac{1}{2} = 16\sqrt{63} c_2$$

$$\frac{63 - 8\sqrt{63}}{2} = 16\sqrt{63} c_2$$

$$c_2 = \frac{63 - 8\sqrt{63}}{2 \cdot 16 \cdot \sqrt{63}} = \frac{\sqrt{63} - 8}{32}$$

$$c_1 = -\frac{1}{2} - c_2 = -\frac{1}{2} - \frac{\sqrt{63} - 8}{32} = \frac{-16 + 8 - \sqrt{63}}{32} = \frac{-8 - \sqrt{63}}{32}$$

$$u(t) = \frac{-8 - \sqrt{63}}{32} e^{(-64+8\sqrt{63})t} + \frac{\sqrt{63} - 8}{32} e^{(-64-8\sqrt{63})t}$$

$$= e^{-64t} \cdot \left(-\frac{1}{4} \cdot e^{8\sqrt{63}t} - \frac{1}{4} e^{-8\sqrt{63}t} + \frac{\sqrt{63}}{32} (e^{-8\sqrt{63}t} - e^{8\sqrt{63}t}) \right)$$

$$= e^{-64t} \cdot \left(-\frac{1}{2} \cdot \cosh 8\sqrt{63}t + \frac{\sqrt{63}}{16} \sinh 8\sqrt{63}t \right)$$

P.05

Grading Guidelines for Exam II, M222, March 09 2016

$$u(t) \sim e^{-64t} e^{8\sqrt{63}t} \left(-\frac{1}{4} + \frac{\sqrt{63}}{32}\right)$$

$$|u| < 0.05$$

$$e^{-64\tau + 8\sqrt{63}\tau} \left| -\frac{1}{4} + \frac{\sqrt{63}}{32} \right| \approx \frac{5}{100}$$

$$e^{(-64 + 8\sqrt{63})\tau} = \frac{5}{100} \cdot \frac{1}{\left| -\frac{1}{4} + \frac{\sqrt{63}}{32} \right|}$$

$$(-64 + 8\sqrt{63})\tau = \ln\left(\frac{5}{100} \cdot \frac{32}{8 - \sqrt{63}}\right)$$

$$\tau = \frac{1}{-64 + 8\sqrt{63}} \cdot \ln \frac{8}{5(8 - \sqrt{63})} \quad [1 \text{ pt}]$$

Problem 6: (a) for an undamped, forced oscillator,

at resonance $u(t) \sim t \cdot \cos \omega_0 t$

bottom panel is a resonance case, [4 pts]

6 period ~ 36 , period ~ 6 , frequency $\omega_0 \sim \frac{2\pi}{6} \text{ s}^{-1}$
(exact answer is 1 s^{-1}) [3 pts]

(b) ω_f for bottom panel $= \omega_0 = 1 \text{ s}^{-1}$ [3 pts] (or $\frac{2\pi}{6} \text{ s}^{-1}$)

for panel (a), the amplitude modulation suggests a

alternatively

[2 pts] beat, which consists of a product of $\frac{\omega_0 - \omega_f}{2}$ & $\frac{\omega_0 + \omega_f}{2}$

$\omega_f \neq \omega_0$

[3 pts]

$\frac{\omega_0 - \omega_f}{2}$ gives rise to the envelope oscillations [3 pts]

$\frac{\omega_0 + \omega_f}{2}$ gives rise to fast oscillations

$$\frac{\omega_0 - \omega_f}{2} \approx 0.1 \Rightarrow \frac{2\pi}{20\pi} = \frac{1}{10}$$

$$\omega_f \sim 0.9 \text{ s}^{-1}$$

[3 pts]

$$\frac{\omega_0 + \omega_f}{2} \approx 1 \Rightarrow \frac{2\pi}{2\pi} = 1$$