

P. 1

Grading Guidelines For Exam 3 April 13 2016

Problem 1: $x^2 y'' + \alpha x y' + y = 0, \quad y_1 = x^{1/2},$

$$\frac{1}{2}(-\frac{1}{2}) + \alpha \cdot \frac{1}{2} + 1 = 0, \quad \frac{3}{4} + \frac{\alpha}{2} = 0, \quad \boxed{\alpha = -\frac{3}{2}} \quad \boxed{4 \text{ pts}}$$

$$r(r-1) - \frac{3}{2}r + 1 = 0, \quad r^2 - \frac{5}{2}r + 1 = 0, \quad 2r^2 - 5r + 2 = 0$$

$$(2r-1)(r-2) = 0. \quad r = \frac{1}{2}, 2, \quad \boxed{y_2 = x^2} \quad \boxed{3 \text{ pts}}$$

Problem 2: $1-x^2=0, \quad x = \pm 1$

$$x=1, \quad \lim_{x \rightarrow 1} \frac{-2x}{1-x^2} (1-x) = \lim_{x \rightarrow 1} \frac{-2x}{1+x} = -1$$

$$\lim_{x \rightarrow 1} \frac{\beta(\beta+1)}{1-x^2} (1-x)^2 = \lim_{x \rightarrow 1} \frac{\beta(\beta+1)}{1+x} (1-x) = 0$$

$x=1$ is a regular singular point $\boxed{4 \text{ pts}}$

$$x=-1, \quad \lim_{x \rightarrow -1} \frac{-2x}{1-x^2} (1+x) = \lim_{x \rightarrow -1} \frac{-2x}{1-x} = 2/2 = 1$$

$$\lim_{x \rightarrow -1} \frac{\beta(\beta+1)}{1-x^2} (1+x)^2 = \lim_{x \rightarrow -1} \frac{\beta(\beta+1)}{1-x} (1+x) = 0$$

$x=-1$ is a regular singular point $\boxed{4 \text{ pts}}$

Problem 3: $y'' + y = 0, \quad -\infty < x < \infty,$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

large n : $(n+2)(n+1) a_{n+2} + a_n = 0, \quad a_{n+2} = -\frac{1}{(n+2)(n+1)} a_n$

P.2

$$x^0: 2 \cdot 1 \cdot a_2 + a_0 = 0, \quad a_2 = -\frac{1}{2 \cdot 1} a_0$$

$$x^1: 3 \cdot 2 \cdot a_3 + a_1 = 0, \quad a_3 = -\frac{1}{3 \cdot 2} a_1$$

$$\text{recurrence relation } a_{n+2} = -\frac{1}{(n+2)(n+1)} a_n, \quad n=0, 1, 2, \dots$$

4 pts

$$y_1 = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4 - \dots \right)$$

4 pts

$$y_2 = a_1 \left(x - \frac{1}{6} x^3 + \frac{1}{5!} x^5 - \dots \right)$$

4 pts

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| < 1 \quad \text{for convergence}$$

$$x^2 < \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+2}} \right| = \lim_{n \rightarrow \infty} (n+2)(n+1), \quad x^2 < \infty, \quad |x| < \infty$$

radius of convergence is ∞

3 pts

Problem 4:

$$(a) F = \frac{s^2 - 9}{s^2 + 6s + 9} = \frac{(s+3)(s-3)}{s(s+3)(s+3)} = \frac{s-3}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$As + 3A + Bs = s - 3, \quad A+B=1, \quad 3A=-3, \quad A=-1, \quad B=2$$

4 pts

$$F = -\frac{1}{s} + \frac{2}{s+3}, \quad \mathcal{L}^{-1}[F] = -1 + 2e^{-3t}$$

4 pts

$$(b) G(s) = \frac{e^{-s}(s-2)}{s^2 + 2s + 2} \quad H = \frac{s-2}{s^2 + 2s + 2} = \frac{s+1-3}{(s+1)^2 + 1}$$

5 pts

$$h(t) = \begin{matrix} e^{-t} \cos t \\ -3e^{-t} \sin t \end{matrix}, \quad \mathcal{L}^{-1}[G] = U_1 h(t-1)$$

4 pts

$$(c) \mathcal{L}[t^2 \sin^2 t] = \mathcal{L}\left[t^2 \left(\frac{1 - \cos 2t}{2}\right)\right] = \mathcal{L}\left[\frac{t^2}{2} - \frac{t^2}{2} \cos 2t\right]$$

3 pts

P.3

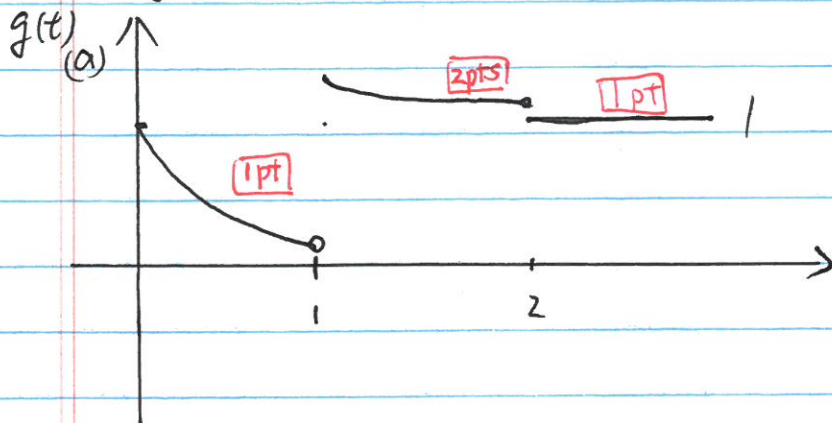
$$\mathcal{L}\left[\frac{t^2}{2}\right] = \frac{2!}{2 \cdot s^3} \quad \mathcal{L}[t^2 \cos 2t] = \left(\frac{s}{s^2+4}\right)'' = \frac{-s \cdot 2s}{(s^2+4)^2} + \frac{1}{s^2+4}$$

$$\mathcal{L}[t^2 \sin^2 t] = \frac{1}{s^3} + \frac{s^2}{(s^2+4)^2} - \frac{1}{2(s^2+2^2)}$$

Problem 5 :

$$g(t) = \begin{cases} e^{-t} & 0 \leq t < 1 \\ e^{3t} + 1 & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases}$$

(b) $g(t) = e^{-t} + (-e^{-t} + e^{-3t} + 1)u_1 + (-e^{-3t})u_2$



(c) $\mathcal{L}[g] = \mathcal{L}[e^{-t} + (-e^{-t} + e^{-3t} + 1)u_1 + (-e^{-3t})u_2]$

$$= \mathcal{L}[e^{-t}] + \mathcal{L}[(e^{-(t-1)} \cdot e^{-1} + e^{-3(t-1)} \cdot e^{-3} + 1)u_1]$$

$$- \mathcal{L}[e^{-3(t-2)} \cdot e^{-2} u_2]$$

$$= \frac{e^{-s}}{s} - e^{-1} \cdot \frac{e^{-s}}{s-1} + e^{-3} \cdot \frac{e^{-s}}{s-3} - e^{-2} \cdot \frac{e^{-2s}}{s-3}$$

P. 4

Problem 6: $y'' + 4y' + 8y = 2u_\pi - 2\delta(t-2\pi)$, $y(0) = 2$; $y'(0) = 0$

$$s^2 Y - s \cdot 2 + 4(sY - 2) + 8Y = 2 \cdot \frac{e^{-\pi s}}{s} - 2e^{-2\pi s} \quad (2 \text{ pts})$$

$$(s^2 + 4s + 8)Y = 2s + 8 + 2 \frac{e^{-\pi s}}{s} - 2e^{-2\pi s}$$

$$Y = \frac{2(s+4)}{(s+2)^2 + 2^2} + \frac{2e^{-\pi s}}{s((s+2)^2 + 2^2)} - \frac{2e^{-2\pi s}}{(s+2)^2 + 2^2} \quad (3 \text{ pts})$$

$$Y = \frac{2(s+2)}{(s+2)^2 + 2^2} + \frac{2 \cdot 2}{(s+2)^2 + 2^2} + \frac{2e^{-\pi s}}{s((s+2)^2 + 2^2)} - \frac{2e^{-2\pi s}}{(s+2)^2 + 2^2}$$

$$y = 2e^{-2t} \cos 2t + 2e^{-2t} \sin 2t + 2 \cdot \mathcal{L}^{-1} \left[\frac{e^{-\pi s}}{s((s+2)^2 + 2^2)} \right] \cdot u_{2\pi} \cdot f(t-2\pi)$$

$f(t) = e^{-2t} \sin 2t$

$$\mathcal{L}^{-1} \left[\frac{1}{s(s^2 + 4s + 8)} \right] = \mathcal{L}^{-1} \left[\frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8} \right] = \frac{1}{8} \mathcal{L}^{-1} \left[\frac{1}{s} \right] + \mathcal{L}^{-1} \left[\frac{-\frac{1}{8}s - \frac{1}{2}}{s^2 + 4s + 8} \right]$$

$$A(s^2 + 4s + 8) + Bs^2 + Cs = 1 \quad = \frac{1}{8} \mathcal{L}^{-1} \left[\frac{1}{s} \right] - \frac{1}{8} \mathcal{L}^{-1} \left[\frac{s+4}{s^2 + 4s + 8} \right]$$

$$8A = 1, A = 1/8,$$

$$B = -\frac{1}{8}, \frac{1}{2} + C = 0, C = -\frac{1}{2}$$

$$= \frac{1}{8} - \frac{1}{8} \cdot (e^{-2t} \cos 2t + e^{-2t} \sin 2t)$$

(5 pts)

$$\therefore y = 2e^{-2t} \cos 2t + 2e^{-2t} \sin 2t + u_\pi g(t-\pi) - u_{2\pi} f(t-2\pi) \quad (5 \text{ pts})$$

$$g(t) = \frac{1}{8} - \frac{1}{8} e^{-2t} (\cos 2t + \sin 2t)$$

$$f(t) = e^{-2t} \sin 2t$$

P. 5

Problem 7: (a) $\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!}$ (2 pts)

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t \quad (2 \text{ pts})$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t \frac{(t-\tau)^2}{2} \sin \tau d\tau. \quad (3 \text{ pts})$$

$$(b) \mathcal{L}\left[\int_0^t (t-\tau)^2 \cos 2\tau d\tau\right] = \frac{2}{s^3} \cdot \frac{s}{s^2+4}$$

(4 pts) (4 pts)