Homework Problems for Week 4-Solution

1. Consider the equation $x^2y'' - 4xy' + 6y = 0$ on $(-\infty, \infty)$
   a) Verify that $y_1 = x^3$ and $y_2 = |x|^3$are linearly independent solutions of the equation on the interval $(-\infty, \infty)$.

   Solution: First check that $y_1 = x^3$is a solution:

$$y_1 = x^3$$
$$(y_1)' = 3x^2$$
$$(y_1)'' = 3 \cdot 2 \cdot x = 6x$$
$$x^2(y_1)'' - 4x(y_1)' + 6y_1 = 0 \quad \leftrightarrow$$
$$x^2 \cdot 6x - 4x \cdot 3x^2 + 6x^3 = 0$$

So, obviously $y_2 = x^3$ solves the equation.

$$y_2 = |x|^3 \quad \leftrightarrow \quad \begin{cases} x^3, if\ x \geq 0 \\ -x^3, if\ x < 0 \end{cases}$$
$$(y_2)' = \begin{cases} 3x^2, if\ x > 0 \\ 0, if\ x = 0 \\ -3x^2, if\ x < 0 \end{cases}$$
$$(y_2)'' = \begin{cases} 6x, if\ x > 0 \\ 0, if\ x = 0 \\ -6x, if\ x < 0 \end{cases}$$

So, obviously $y_2 = |x|^3$ solves the equation too.

$$C_1y_1 + C_2y_2 = 0$$

$$\begin{cases} C_1x^3 + C_2x^3 = 0 \leftrightarrow x^2(C_1 + C_2) = 0 \leftrightarrow C_1 = -C_2 \\ \\ C_1x^3 - C_2x^3 = 0 \leftrightarrow x^2(C_1 - C_2) = 0 \leftrightarrow C_1 = C_2 \end{cases}$$
$$then\ C_1 = C_2 = 0$$

$So\ y_1, y_2\ are\ linearly\ independent.$

b) Show that $W(y_1, y_2) = 0$ for every real x. Does this result violate theorem 3.2.4? Explain.

$$W(y_1, y_2) = \begin{vmatrix} x^3 & \pm x^3 \\ 3x^2 & \pm 3x^2 \end{vmatrix} = \pm 3x^5 - (\pm 3x^5) \equiv 0$$

But this result doesn't violate the theorem, because

$P(x) = -\frac{4}{x}, and\ Q(x) = \frac{6}{x^2}$ are not continuous on $((-\infty, \infty)$

c) Verify that $Y_1 = \mathrm{x}^2$ and $Y_2 = x^3$are linearly independent solutions of the equation on the interval $(-\infty, \infty)$.

$$W(Y_1, Y_2) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4 \neq 0, if\ x \neq 0,$$

$$so\ Y_1, Y_2 are\ linearly\ independent$$

d) Both combinations $C_1y_1 + C_2y_2$ and $B_1Y_1 + B_2Y_2$ are solutions of the given equation. (Why? Explain.)

$$C_1y_1 + C_2y_2\ and\ B_1Y_1 + B_2Y_2$$

$$are\ solutions\ by\ Theorem\ 3.2.2\ (Principle\ of\ Superposition)$$

Discuss whether one, both or neither of these combinations is a general solution of the equation on $(-\infty, \infty)$.

Neither of these combinations is a general solution of the equation on $(-\infty, \infty)$.

2. Find a second order linear equation with constant coefficients that has a solution $y = e^x cos3x$.

$$y = e^x cos3x\ means\ that\ C.P.has\ the\ complex\ roots$$

$$r_{1,2} = 1 \pm 3i$$

And can be factored as

$$(r - (1 + 3i)) \cdot (r - (1 - 3i)) = 0$$

$$\mathrm{r}^2 - 2r + 10 = 0$$

So, the corresponding diff.equation is

$$y'' - 2y' + 10y = 0$$