1. Consider the equation $x^2 y'' - 4xy' + 6y = 0$ on $(-\infty, \infty)$

a) Verify that $y_1 = x^3$ and $y_2 = |x|^3$ are linearly independent solutions of the equation on the interval $(-\infty, \infty)$.

Solution: First check that $y_1 = x^3$ is a solution:

$$y_1 = x^3$$

$$(y_1)' = 3x^2$$

$$(y_1)'' = 3 \cdot 2 \cdot x = 6x$$

$$x^2 (y_1)'' - 4x (y_1)' + 6y_1 = 0 \iff$$

$$x^2 \cdot 6x - 4x \cdot 3x^2 + 6x^3 = 0$$

So, obviously $y_1 = x^3$ solves the equation.

$$y_2 = |x|^3 \iff \begin{cases} 
  x^3, & \text{if } x \geq 0 \\
  -x^3, & \text{if } x < 0 
\end{cases}$$

$$(y_2)' = \begin{cases} 
  3x^2, & \text{if } x > 0 \\
  0, & \text{if } x = 0 \\
  -3x^2, & \text{if } x < 0 
\end{cases}$$

$$(y_2)'' = \begin{cases} 
  6x, & \text{if } x > 0 \\
  0, & \text{if } x = 0 \\
  -6x, & \text{if } x < 0 
\end{cases}$$

So, obviously $y_2 = |x|^3$ solves the equation too.

$$C_1y_1 + C_2y_2 = 0$$

$$\begin{align*}
C_1 x^3 + C_2 x^3 &= 0 \iff x^2 (C_1 + C_2) = 0 \iff C_1 = -C_2 \\
C_1 x^3 - C_2 x^3 &= 0 \iff x^2 (C_1 - C_2) = 0 \iff C_1 = C_2 \\
\text{then } C_1 &= C_2 = 0
\end{align*}$$

So $y_1, y_2$ are linearly independent.
b) Show that $W(y_1, y_2) = 0$ for every real $x$. Does this result violate theorem 3.2.4? Explain.

\[
W(y_1, y_2) = \begin{vmatrix} x^3 & \pm x^3 \\ 3x^2 & \pm 3x^2 \end{vmatrix} = \pm 3x^5 - (\pm 3x^5) \equiv 0
\]

But this result doesn’t violate the theorem, because $P(x) = -\frac{4}{x}$, and $Q(x) = \frac{6}{x^2}$ are not continuous on $((-\infty, \infty))$

c) Verify that $Y_1 = x^2$ and $Y_2 = x^3$ are linearly independent solutions of the equation on the interval $(-\infty, \infty)$.

\[
W(Y_1, Y_2) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4 \neq 0, \text{ if } x \neq 0,
\]

so $Y_1, Y_2$ are linearly independent

d) Both combinations $C_1 y_1 + C_2 y_2$ and $B_1 Y_1 + B_2 Y_2$ are solutions of the given equation. (Why? Explain.)

$C_1 y_1 + C_2 y_2$ and $B_1 Y_1 + B_2 Y_2$

are solutions by Theorem 3.2.2 (Principle of Superposition)

Discuss whether one, both or neither of these combinations is a general solution of the equation on $(-\infty, \infty)$.

Neither of these combinations is a general solution of the equation on $(-\infty, \infty)$.

2. Find a second order linear equation with constant coefficients that has a solution $y = e^x \cos 3x$.

$y = e^x \cos 3x$ means that C.P. has the complex roots $r_{1,2} = 1 \pm 3i$

And can be factored as

\[
(r - (1 + 3i)) \cdot (r - (1 - 3i)) = 0
\]

\[
r^2 - 2r + 10 = 0
\]

So, the corresponding diff.equation is

\[
y'' - 2y' + 10y = 0
\]