Solutions to Homework for Week 7 M222 Spring '16

Prob 1 : $4y'' + 12y' + 9y = 37\sin t$, $y(0) = 1$, $y'(0) = 0$

$4r^2 + 12r + 9 = 0$, $(2r+3)^2 = 0$, $r = -\frac{3}{2}, -\frac{3}{2}$

$y = c_1 e^{-\frac{3}{2}t} + c_2 t e^{-\frac{3}{2}t} + Y$

$Y = A\cos t + B\sin t$, $Y' = -A\sin t + B\cos t$, $Y'' = -A\cos t - B\sin t$

$4(-A\cos t - B\sin t) + 12(-A\sin t + B\cos t) + 9(A\cos t + B\sin t) = 37\sin t$

$\cos t$: $-4A + 12B + 9A = 0$

$\sin t$: $-4B - 12A + 9B = 0 + 37$

$5A + 12 \cdot B = 0$ $\quad B = -\frac{5}{12}A$

$-12A + 5B = 37$

$-12A + -\frac{25}{12}A = 0$ $\quad -\frac{169}{12}A = 37$, $A = -\frac{37 \times 12}{169}$

$A = -\frac{444}{169}$, $B = \frac{5}{12} \cdot \frac{444}{169} = \frac{185}{169}$

$y = c_1 e^{-\frac{3}{2}t} + c_2 t e^{-\frac{3}{2}t} + \left(-\frac{444}{169}\cos t + \frac{185}{169}\sin t\right)$

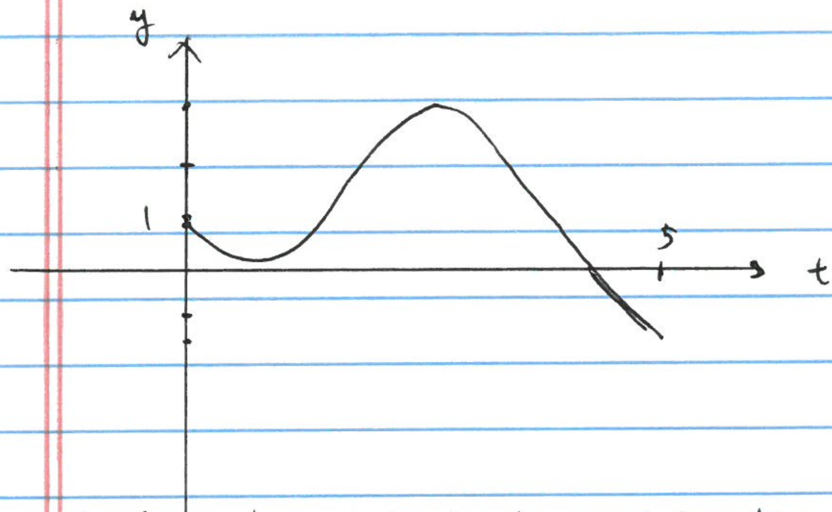
$y'(t) = -\frac{3}{2}c_1 e^{-\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t} + \left(-\frac{3}{2}\right)c_2 t e^{-\frac{3}{2}t} + \left(+\frac{444}{169}\sin t + \frac{185}{169}\cos t\right)$

$y(0) = 1 = c_1 - \frac{444}{169}$, $c_1 = \frac{444}{169} + 1 = \frac{613}{169}$

$y'(0) = -\frac{3}{2}c_1 + c_2 + \frac{185}{169} = 0$ $\quad c_2 = \frac{3}{2}c_1 - \frac{185}{169} = \frac{507 - 370}{338} = \frac{237}{338}$

$y(t) = \frac{613}{169}e^{-\frac{3}{2}t} + \frac{237}{338}te^{-\frac{3}{2}t} + \left(-\frac{444}{169}\cos t + \frac{185}{169}\sin t\right)$

P. 2



(b) from the plot it shows that $y=0$ at around $t \sim 4.2$

(c) $y'=0$, from the plot it can be observed that a local minimum $y \sim 0.1$ is around $t \sim 0.5$, a local maximum is around $t \sim 2.75$ $(y \sim 2.9)$

Prob 2 :

$$y_1 = e^{-\delta x^2/2}, \quad y_1' = -\delta x \cdot e^{-\delta x^2/2}, \quad y_1'' = -\delta \cdot e^{-\delta x^2/2} + \delta^2 x^2 \cdot e^{-\delta x^2/2}$$

$$y_1'' + \delta(x y_1' + y_1) = (-\delta + \delta^2 x^2) e^{-\delta x^2/2} + \delta(-\delta x^2 e^{-\delta x^2/2} + e^{-\delta x^2/2})$$
$$= 0, \quad y_1 \text{ is a solution.}$$

$$y_2 = v \cdot y_1, \quad y_2' = v' y_1 + v y_1', \quad y_2'' = v'' y_1 + 2v' y_1' + v y_1''$$

$$v'' y_1 + 2v' y_1' + v y_1'' + \delta \cdot x \cdot (v' y_1 + v y_1') + \delta \cdot v y_1 = 0$$
$$v'' y_1 + 2v' y_1' + \delta x \cdot v' y_1 = 0$$

$$v'' e^{-\delta x^2/2} + 2v' \cdot (-\delta x e^{-\delta x^2/2}) + \delta \cdot x \cdot v' \cdot e^{-\delta x^2/2} = 0$$
$$v'' - 2\delta x v' + \delta x v' = 0$$
$$v'' = \delta x v', \quad \frac{v''}{v'} = \delta x, \quad \ln v' = \frac{\delta x^2}{2}$$
$$v' = e^{\delta x^2/2}, \quad v = \int e^{\delta x^2/2} dx$$

P. 3

$$y_2 = \left(\int e^{\delta x^2/2}\,dx\right)\cdot e^{-\delta x^2/2} \rightarrow \text{ general solution } y = c_1 y_1 + c_2 y_2$$

Prob 3 :

$$y'' + 5y' + 6y = 2t^5 + (3+2t)e^{-2t} + 4te^{-3t}\cdot\cos 2t$$

$$r^2 + 5r + 6 = 0, \quad (r+3)(r+2) = 0, \quad r = -3, -2$$

$$y_1 = e^{-3t}, \quad y_2 = e^{-2t}$$

$$Y = At^5 + Bt^4 + Ct^3 + Dt^2 + Et + F + t(G + Ht)e^{-2t} + (I + Jt)e^{-3t}(K\cos 2t + L\sin 2t)$$

Prob 4 :

$$y'' - 2y' + y = \frac{e^t}{4+t^2}, \quad y(0) = 0, \quad y'(0) = 0$$

$$r^2 - 2r + 1 = 0, \quad r = 1, 1 \qquad y_1 = e^t, \quad y_2 = te^t$$

$$Y = \left(\int \frac{-\frac{e^t}{4+t^2}\cdot te^t}{W(y_1, y_2)}\right) e^t + \left(\int \frac{\frac{e^t}{4+t^2}\cdot e^t}{W(y_1, y_2)}\right) te^t$$

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$$

$$Y = \left(\int \frac{-\frac{te^{2t}}{t^2+4}}{e^{2t}}\right) e^t + \left(\int \frac{\frac{e^{2t}}{t^2+4}}{e^{2t}}\right) te^t$$

$$= -\left(\int \frac{t}{t^2+4}\right) e^t + \left(\int \frac{1}{t^2+4}\right) te^t$$

$$Y = -\frac{1}{2}\ln(t^2+4)\,e^t + \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\cdot te^t$$

$$y = c_1 e^t + c_2 te^t + Y, \quad y' = c_1 e^t + c_2 e^t + c_2 te^t + Y'$$

P. 4

$y(0) = 0 \Rightarrow C_1 + Y(0) = 0 \qquad Y(0) = -\frac{1}{2}\ln 4 \,, \quad \boxed{C_1 = \ln 2}$

$y'(0) = 0, \qquad C_1 + C_2 + Y'(0) = 0$

$$Y' = -\frac{t}{t^2+4}e^t - \frac{1}{2}\ln(t^2+4)e^t + \frac{1}{2}\cdot\frac{1}{1+\left(\frac{t}{2}\right)^2}\,te^t$$

$$+\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\left(e^t + te^t\right)$$

$Y'(0) = -\frac{1}{2}\ln 4 \qquad C_1 + C_2 + Y'(0) = 0$

$\ln 2 + C_2 - \frac{1}{2}\ln 4 = 0$

$\boxed{C_2 = 0}$

$$y = \ln 2\, e^t + \left(-\frac{1}{2}\ln(t^2+4)\,e^t + \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\cdot te^{-t}\right)$$