Solutions to HW Chapter 5

Problem 1: (a) $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$, $\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n \cdot x^n} \right| < 1$

$$\lim_{n\to\infty} \left| \frac{\frac{n+1}{2^{n+1}} x^{n+1}}{\frac{n}{2^n} x^n} \right| < 1, \quad \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{1}{2} \cdot x \right| < 1$$

$$\left| \frac{x}{2} \right| < 1 \qquad |x| < 2$$

radius of convergence is 2

(b) $\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$ $\quad \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}} x^{n+1}}{\frac{n!}{n^n} x^n} \right| < 1$

$$\lim_{n\to\infty} \left| (n+1) \frac{n^n}{(n+1)^{n+1}} x \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} x \right| < 1 \qquad |x| < \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = e$$

radius of convergence is e

Problem 2: (a) $(x+1)y'' - x^2 y' + 3y = 0$

$$y'' - \frac{x^2}{x+1} y' + \frac{3}{x+1} y = 0$$

$x+1=0$, $x=-1$ is a singular point

$$\lim_{x\to -1} (x+1) \frac{x^2}{x+1} = 1, \quad \lim_{x\to -1} (x+1)^2 \frac{3}{x+1} = 0$$

$x=-1$ is a regular singular point

(b) $(x^2+x)y'' + 3y' - 6xy = 0$

$$y'' + \frac{3}{x(x+1)} y' - \frac{6x}{x(x+1)} y = 0$$

$x=0$, $x=-1$ are singular points

$$x=0, \quad \lim_{x\to 0} x \frac{3}{x(x+1)} = 3, \quad \lim_{x\to 0} x^2 \frac{6x}{x(x+1)} = 0$$

$$x=-1, \quad \lim_{x\to -1} (x+1) \frac{3}{x(x+1)} = -3, \quad \lim_{x\to -1} (x+1)^2 \frac{6x}{x(x+1)} = 0$$

both $x=0$ & $x=-1$ are regular singular points

(c) $(x^2-x-2)y'' + (x+1)y' - (x-2)y = 0$

$$y'' + \frac{x+1}{(x-2)(x+1)}y' - \frac{(x-2)}{(x-2)(x+1)}y = 0$$

$x=-1$, $x=2$ are singular points

$$\lim_{x\to -1}(x+1)\frac{x+1}{(x-2)(x+1)} = 0, \quad \lim_{x\to -1}(x+1)^2\frac{x-2}{(x-2)(x+1)} = 0$$

$$\lim_{x\to 2}(x-2)\frac{x+1}{(x-2)(x+1)} = 1, \quad \lim_{x\to 2}(x-2)^2\frac{x-2}{(x-2)(x+1)} = 0$$

both -1 & 2 are regular singular points

Problem 3: (a) $y' = xy$, $\quad y = \sum_{n=0}^{\infty} a_n x^n$, $\quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

x^0 terms: $1\,a_1 = 0$ $\quad a_1 = 0$

x^1 terms: $2\cdot a_2 - a_0 = 0$ $\quad a_2 = \frac{1}{2}a_0$

x^2 terms: $3a_3 - a_1 = 0$ $\quad a_3 = \frac{1}{3}a_1 = 0$

$\vdots$

x^n terms: $(n+1)a_{n+1} - a_{n-1} = 0$ $\quad a_{n+1} = \frac{1}{n+1}a_{n-1}$

$a_4 = \frac{1}{4}a_2 = \frac{1}{4}\cdot\frac{1}{2}a_0$

First four terms: $y \sim a_0 + \frac{a_0}{2}x^2 + \frac{1}{4\cdot 2}x^4 + \frac{1}{6\cdot 4\cdot 2}x^6 + \cdots$

(b) $9x^2y'' + 9x^2y' + 2y = 0$, $\quad x=0$ is a regular singular point

$$p_0 = \lim_{x\to 0} x \cdot \frac{9x^2}{9x^2} = 0, \quad q_0 = \lim_{x\to 0} x^2 \cdot \frac{2}{9x^2} = \frac{2}{9}$$

$$r(r-1) + 0 \cdot r + \frac{2}{9} = 0$$

$$r^2 - r + \frac{2}{9} = 0 \qquad \left(r - \frac{2}{3}\right)\left(r - \frac{1}{3}\right) = 0 \quad r = \frac{1}{3}, \frac{2}{3}$$

$$y = x^{1/3} \sum_{n=0}^{\infty} a_n x^n \quad \text{or} \quad y = x^{2/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y' = \sum_{n=0}^{\infty} (n+1/3) a_n x^{n-2/3} \quad \text{or} \quad y' = \sum_{n=0}^{\infty} (n+2/3) b_n x^{n-1/3}$$

$$y'' = \sum_{n=0}^{\infty} \left(n+\frac{1}{3}\right)(n-2/3) a_n x^{n-5/3} \quad \text{or} \quad y'' = \sum_{n=0}^{\infty} (n+2/3)(n-1/3) b_n x^{n-4/3}$$

$$9x^2y'' + 9x^2y' + 2y = 0$$

$r = \frac{1}{3}$

$$\sum_{n=0}^{\infty} 9(n+1/3)(n-2/3) a_n x^{n+1/3} + \sum_{n=0}^{\infty} 9(n+1/3) a_n x^{n+4/3} + \sum_{n=0}^{\infty} 2a_n x^{n+1/3} = 0$$

$$\sum_{n=0}^{\infty} 9(n+1/3)(n-2/3) a_n x^{n+1/3} + \sum_{n=1}^{\infty} 9(n-2/3) a_{n-1} x^{n+1/3} + \sum_{n=0}^{\infty} 2a_n x^{n+1/3} = 0$$

$$9\left(n^2 - \frac{n}{3} - \frac{2}{9}\right) a_n + 9\left(n - \frac{2}{3}\right) a_{n-1} + 2a_n = 0$$

$$9\left(n^2 - \frac{n}{3}\right) a_n + 9\left(n - \frac{2}{3}\right) a_{n-1} = 0$$

$$a_n = -\frac{n-2/3}{n(n-1/3)} a_{n-1}, \qquad n \geq 1$$

$r = \frac{2}{3}$

$$\sum_{n=0}^{\infty} 9(n+2/3)(n-1/3) b_n x^{n+2/3} + \sum_{n=0}^{\infty} 9(n+2/3) b_n x^{n+5/3} + \sum_{n=0}^{\infty} 2b_n x^{n+2/3} = 0$$

$$9(n+2/3)(n-1/3) b_n + 9(n-1/3) b_{n-1} + 2b_n = 0$$

$$9\left(n^2 + \frac{n}{3} - \frac{2}{9}\right) b_n + 9(n-1/3) b_{n-1} + 2b_n = 0$$

$$9 \cdot n\left(n+\frac{1}{3}\right) b_n = -9\left(n - \frac{1}{3}\right) b_{n-1}$$

$$b_n = -\frac{n-1/3}{n(n+1/3)} b_{n-1}, \qquad n \geq 1$$

(c) $y'' - xy' - y = 0, \quad y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

x^0: $\quad 2\cdot 1\cdot a_2 - a_0 = 0, \qquad a_2 = \frac{1}{2} a_0$

x^1: $\quad 3\cdot 2\cdot a_3 - a_1 - a_1 = 0, \qquad a_3 = \frac{1}{3} a_1$

x^2: $\quad 4\cdot 3\cdot a_4 - 2a_2 - a_2 = 0, \qquad a_4 = \frac{1}{4} a_2$

x^n: $\quad (n+2)(n+1) a_{n+2} - n a_n - a_n = 0, \qquad a_{n+2} = \frac{1}{n+2} a_n$

$$y \sim a_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{2}\cdot\frac{1}{4} x^4 + \cdots\right) + a_1 \left(x + \frac{1}{3} x^3 + \frac{1}{3}\cdot\frac{1}{5} x^5 + \cdots\right)$$

Problem 4: $\quad y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$

$$y'' - 2xy' + \lambda y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0,$$

x^0: $\quad 2\cdot 1\cdot a_2 + \lambda a_0 = 0, \quad a_2 = -\frac{\lambda}{2} a_0$

x^1: $\quad 3\cdot 2\cdot a_3 - 2a_1 + \lambda a_1 = 0, \quad a_3 = -\frac{(\lambda-2)}{3\cdot 2} a_1$

x^2: $\quad 4\cdot 3\cdot a_4 - 2\cdot 2\cdot a_2 + \lambda a_2 = 0, \quad a_4 = -\frac{\lambda-4}{4\cdot 3} a_2$

x^n: $\quad (n+2)(n+1) a_{n+2} - 2n a_n + \lambda a_n = 0, \quad a_{n+2} = -\frac{\lambda - 2n}{(n+2)(n+1)} \cdot a_n$

$$y_1 = a_0 \left(1 - \frac{\lambda}{2} x^2 + \frac{(\lambda-4)\lambda}{4!} x^4 - \frac{(\lambda-8)(\lambda-4)\lambda}{6!} x^6 + \cdots\right)$$

$$y_2 = a_1 \left(x - \frac{\lambda-2}{3\cdot 2} x^3 + \frac{(\lambda-6)(\lambda-2)}{5!} x^5 - \frac{(\lambda-10)(\lambda-6)(\lambda-2)}{7!} x^7 - \cdots\right)$$

$\lambda = 0, \quad y_1 = 1$

$\lambda = 2, \quad y_2 = x$

$\lambda = 4 \quad y_1 = 1 - 2x^2$