Solutions for Homework Problems (Chapter 6)

Problem 1: (a) $a(t) = te^{t}\sin t$

$$\mathcal{L}[e^{t}\sin t] = \frac{1}{(s-1)^2+1^2} = F(s)$$

$$\mathcal{L}[-t\cdot f(t)] = F'(s)$$

$$\mathcal{L}[t\cdot f(t)] = -F'(s) = \frac{2(s-1)}{((s-1)^2+1^2)^2}.$$

(b) $b(t) = 2t\cos^2 t \qquad 2\cos^2 t = \cos 2t + 1$

$$b(t) = t\cdot(\cos 2t + 1) = t\cos 2t + t$$

$$\mathcal{L}[b] = \mathcal{L}[t\cos 2t + t] = -\left(\frac{s}{s^2+2^2}\right)' + \frac{1}{s^2}$$

$$\mathcal{L}[b] = \frac{-1}{s^2+2^2} + \frac{s\cdot 2s}{(s^2+2^2)^2} + \frac{1}{s^2}$$

(c) $c(t) = t^3 e^{-3t}$,

$$\mathcal{L}[e^{-3t}] = \frac{1}{s+3} = F(s)$$

$$\mathcal{L}[t^3 e^{-3t}] = -F^{(3)}(s) = -\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = \frac{6}{(s+3)^4}$$

Problem 2: (a) $\mathcal{L}[f] = \frac{s}{(s-3)^3} = \cancel{\frac{A}{s-3}+\frac{\cdots}{(s-3)^2}} \quad \frac{s-3+3}{(s-3)^3} = \frac{1}{(s-3)^2} + \frac{3}{(s-3)^3}$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right] = -t\cdot e^{3t}$$

$$\mathcal{L}^{-1}\left[\frac{3}{(s-3)^3}\right] = \frac{3}{2}\mathcal{L}^{-1}\left[\frac{2}{(s-3)^3}\right] = \frac{3}{2}t^2 e^{3t}$$

$$\therefore f(t) = -te^{3t} + \frac{3}{2}t^2 e^{3t}$$

(b) $\mathcal{L}[g] = \frac{s+1}{s^2+2s+10} = \frac{s+1}{(s+1)^2+3^2} \Rightarrow g = e^{-t}\cdot\cos 3t$

(c) $\mathcal{L}[h] = \frac{s^2+4s-15}{(s-1)(s^2+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9}$

$$A(s^2+9) + (Bs+c)(s-1) = s^2+4s-15$$

$$A+B = 1$$
$$C-B = 4 \qquad A+c=5$$
$$9A-c = -15$$

$$10A=-10, \quad A=-1, \quad c=6, \quad B=2$$

$$\mathcal{L}[h] = \frac{-1}{s-1} + \frac{2s+6}{s^2+9} = -\frac{1}{s-1} + 2\cdot\frac{s+3}{s^2+3^2} = -\frac{1}{s-1} + 2\cdot\frac{s}{s^2+3^2} + 2\cdot\frac{3}{s^2+3^2}$$

$$h(t) = -e^t + 2\cdot\cos 3t + 2\cdot\sin 3t$$

Problem 3: $y''+3y'+2y = -5\sin t + 5\cos t, \quad y(0)=5, \quad y'(0)=-3$

$$s^2Y - s\cdot 5 - (-3) + 3\cdot(sY-5) + 2Y = -5\cdot\frac{1}{s^2+1} + 5\cdot\frac{s}{s^2+1}$$

$$(s^2+3s+2)Y = 5s-3+15 - 5\frac{1}{s^2+1} + 5\cdot\frac{s}{s^2+1},$$
$$= 5s+12 - 5\cdot\frac{1}{s^2+1} + 5\cdot\frac{s}{s^2+1}$$

$$Y = \frac{5s+12}{s^2+3s+2} - \frac{5(1-s)}{(s^2+1)(s^2+3s+2)}$$

$$Y = \frac{5s+12}{(s+2)(s+1)} - 5\cdot\frac{1-s}{(s^2+1)(s+2)(s+1)}$$

$$\frac{A}{s+2} + \frac{B}{s+1} = \frac{5s+12}{(s+2)(s+1)}, \qquad A(s+1)+B(s+2) = 5s+12$$

$$s=-1, \quad B = -5+12 = 7$$
$$s=-2, \quad -A = -10+12 = 2, \quad A=-2$$

$$\frac{cs+D}{s^2+1} + \frac{E}{s+2} + \frac{F}{s+1} = \frac{1-s}{(s^2+1)(s+2)(s+1)},$$

$$(cs+D)(s+2)(s+1) + E(s^2+1)(s+1) + F(s^2+1)(s+2) = 1-s$$

$$s=-2, \quad E\cdot 5\cdot(-1) = 3, \quad E = -3/5$$

$$s=-1, \quad F\cdot 2\cdot 1 = 2, \quad F = 1$$

P. 03

$s=0, \quad D\cdot 2\cdot 1 - \frac{3}{5}(1)(1) + 1\cdot 1\cdot 2 = 1$

$2D - \frac{3}{5} + 2 = 1, \quad 2D = \frac{3}{5} - 1 = \frac{-2}{5}, \quad D = -\frac{1}{5}$

$s=1, \quad \left(C-\frac{1}{5}\right)\cdot 3\cdot 2 - \frac{3}{5}\cdot 2\cdot 2 + 1\cdot 2\cdot 3 = 0$

$6\left(C-\frac{1}{5}\right) = -6 + \frac{12}{5} = \frac{-30+12}{5} = \frac{-18}{5}, \quad C-\frac{1}{5} = -\frac{3}{5}, \quad C = \frac{-2}{5}$

$$Y = \frac{-2}{s+2} + \frac{7}{s+1} - 5\cdot\frac{\frac{-2}{5}s - \frac{1}{5}}{s^2+1} - \frac{5\cdot\frac{-3}{5}}{s+2} - \frac{5\cdot 1}{s+1}$$

$$= -\frac{2}{s+2} + \frac{7}{s+1} + \frac{2s+1}{s^2+1} + \frac{3}{s+2} - \frac{5}{s+1}$$

$$= -\frac{2}{s+2} + \frac{2}{s+1} + 2\cdot\frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{3}{s+2}$$

$y(t) = -2e^{-2t} + 2\cdot e^{-t} + 2\cdot\cos t + \sin t + 3\cdot e^{-2t}$

$\boxed{y(t) = e^{-2t} + 2e^{-t} + 2\cos t + \sin t}$

Problem 4: $y'' + y = 4u_\pi(t), \; y(0) = 2, \; y'(0) = 4$

$s^2 Y - s\cdot 2 - 4 + Y = 4\cdot\frac{e^{-\pi s}}{s}$

$(s^2+1)Y = 2s+4 + 4\cdot\frac{e^{-\pi s}}{s}$

$Y = \frac{2s+4}{s^2+1} + \frac{4\cdot e^{-\pi s}}{s(s^2+1)}$

$\frac{4}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}, \quad A(s^2+1) + Bs^2 + Cs = 4$

$s^2: \; A + B = 0$

$s: \; C = 0$

$s^0: \; A = 4$

$\frac{4}{s(s^2+1)} = \frac{4}{s} - \frac{4s}{s^2+1}, \quad \boxed{y(t) = 2\cdot\cos t + 4\cdot\sin t + 4(1-\cos(t-\pi))u_\pi}$