Solutions to Homework Problems for Chapter 7

Problem 1: $ay'' + by' + cy = 0$

(a) $x_1 = y$, $x_2 = y'$, $x_1' = x_2$

$x_2' = y''$, $ax_2' + bx_2 + cx_1 = 0$, $ax_2' = -bx_2 - cx_1$

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{b}{a}x_2 - \frac{c}{a}x_1, \end{cases} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(b) $$\det\begin{pmatrix} 0-\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a}-\lambda \end{pmatrix} = 0$$

$$(-\lambda)\left(-\frac{b}{a}-\lambda\right) + \frac{c}{a} = 0$$

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0, \qquad a\lambda^2 + b\lambda + c = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Problem 2: (a) $y'' - 4y' + 6y = 0$

$x_1 = y$, $x_2 = y'$, $x_2' = y'' = 4y' - 6y = 4x_2 - 6x_1$

$$\begin{cases} x_1' = x_2 \\ x_2' = -6x_1 + 4x_2 \end{cases}$$

(b) $y''' + 5y'' + 7y' - 9y = 5t^3 \cdot \cos 2t$

$x_1 = y$, $x_2 = y'$, $x_3 = y''$, $x_4' = y^{(3)} = -5y'' - 7y' + 9y + 5t^3\cos$
$= -5x_3 - 7x_2 + 9x_1 + 5t^3\cos$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = -5x_3 - 7x_2 + 9x_1 + 5t^3\cos 2t \end{cases}$$

Problem 3: $$X(t) = c_1 e^{+5t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Problem 4: $$x' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x \qquad \begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = 0, \qquad (\lambda-1)(\lambda+3) + 5 = 0$$

$$\lambda^2 + 2\lambda - 3 + 5 = 0, \quad (\lambda+1)^2 + 1 = 0, \quad \lambda = -1 \pm i$$

$\lambda = -1+i$, $\begin{pmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(2-i)x_1 - 5x_2 = 0$, $\quad \frac{x_1}{x_2} = \frac{5}{2-i} = \frac{5(2+i)}{4+1} = 2+i$

eigenvector $\Rightarrow \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

for $\lambda = -1-i$, eigenvector $\rightarrow \begin{pmatrix} 2-i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$x = c_1 e^{-t} \cdot \left(\cos t \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$+ c_2 e^{-t} \cdot \left(\sin t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

as $t \to \infty$, $\quad X \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.