

Solutions to Homework Problems for Chapter 10

Problem 1: $x^2 y'' - x y' + \lambda y = 0$, $y(1) = 0$, $y(L) = 0$, $L > 1$

$$r(r-1) - r + \lambda = 0$$

$$r^2 - 2r + \lambda = 0, \quad (r-1)^2 + \lambda - 1 = 0, \quad (r-1)^2 = 1 - \lambda, \quad r = 1 \pm \sqrt{1-\lambda}$$

$$\lambda = 1 \quad r = 1, 1 \quad y = c_1 x + c_2 x \ln x$$

$$y(1) = 0 = c_1 + c_2 \cdot 0, \quad c_1 = 0$$

$$y(L) = 0 = c_1 L + c_2 L \ln L = 0, \quad c_2 = 0$$

$\lambda = 1$ is not an eigenvalue.

$$\lambda > 1 \quad r = 1 \pm \sqrt{\lambda-1} i, \quad \lambda-1 > 0$$

$$y = c_1 x \cos(\sqrt{\lambda-1} \ln x) + c_2 x \sin(\sqrt{\lambda-1} \ln x)$$

$$y(1) = c_1 \cdot 1 \cdot \cos(0) + c_2 \cdot 1 \cdot \sin(0) = 0, \quad c_1 = 0$$

$$y(L) = c_2 \cdot L \cdot \sin(\sqrt{\lambda-1} \ln L) = 0, \quad \sqrt{\lambda-1} \ln L = n\pi$$

$$\lambda - 1 = \left(\frac{n\pi}{\ln L}\right)^2, \quad \lambda = 1 + \left(\frac{n\pi}{\ln L}\right)^2$$

$$\lambda < 1, \quad r = 1 \pm \sqrt{1-\lambda}, \quad y = c_1 x^{1+\sqrt{1-\lambda}} + c_2 x^{1-\sqrt{1-\lambda}}$$

$$y(1) = c_1 + c_2 = 0$$

$$y(L) = c_1 L^{1+\sqrt{1-\lambda}} + c_2 L^{1-\sqrt{1-\lambda}} = 0$$

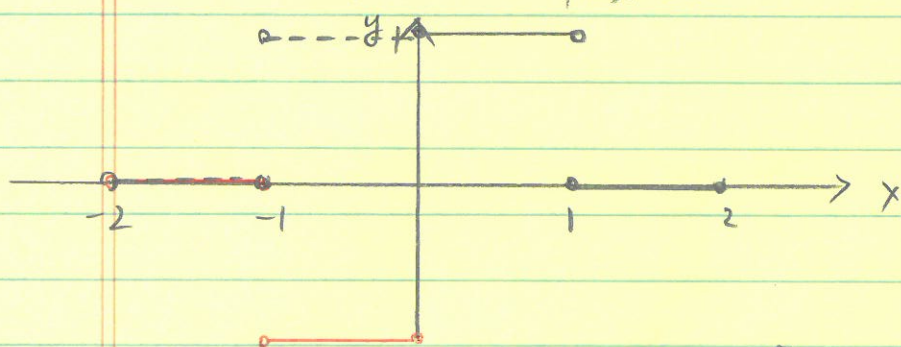
$$c_1 = c_2 = 0$$

$\lambda < 1$ is not an eigenvalue.

Therefore $\lambda_n = 1 + \left(\frac{n\pi}{\ln L}\right)^2$,

$$y_n = x \sin\left(\frac{n\pi \ln x}{\ln L}\right)$$

Problem 2: (a) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$



$$2L = 4, \quad L = 2$$

• even extension is in dashed line,

• odd extension is in solid line

even extension: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$,

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx$$

$$= \frac{2}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx$$

P.2

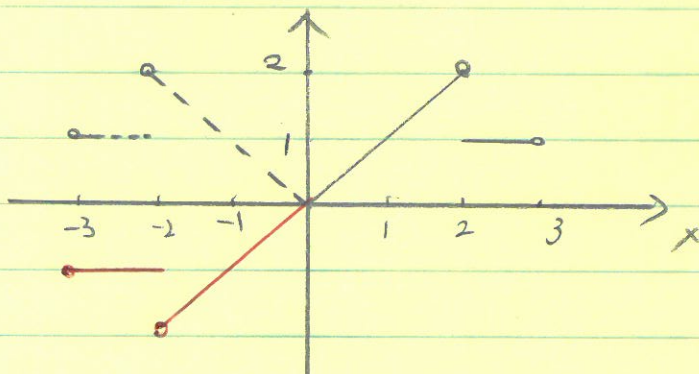
$$a_n = \int_0^1 1 \cdot \cos \frac{n\pi x}{2} dx + \int_1^2 0 \cdot \cos \frac{n\pi x}{2} dx = \frac{1}{\frac{n\pi}{2}} \sin \frac{n\pi x}{2} \Big|_0^1$$

$$\text{When } n=0, \quad a_0 = \int_0^1 1 dx + \int_1^2 0 dx = 1$$

$$\text{odd extension} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}, \quad b_n = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx$$

$$b_n = \int_0^1 \sin \frac{n\pi x}{2} dx + \int_1^2 0 dx = -\frac{1}{\frac{n\pi}{2}} \cos \frac{n\pi x}{2} \Big|_0^1 = -\frac{2}{n\pi} \left(\cos \frac{n\pi}{2} - 1 \right)$$

$$(b) f(x) = \begin{cases} x & 0 \leq x < 2 \\ 1 & 2 \leq x < 3 \end{cases}$$



$$2L=6, L=3$$

$$\text{even extension:} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3}$$

$$a_n = \frac{1}{3} \int_{-3}^3 f(x) \cos \frac{n\pi x}{3} dx = \frac{2}{3} \left(\int_0^2 x \cos \frac{n\pi x}{3} dx + \int_2^3 \cos \frac{n\pi x}{3} dx \right)$$

$$\int x \cos \frac{n\pi x}{3} dx = x \frac{1}{\frac{n\pi}{3}} \sin \frac{n\pi x}{3} - \int \frac{1}{\frac{n\pi}{3}} \sin \frac{n\pi x}{3} dx$$

$$= \frac{3}{n\pi} x \sin \frac{n\pi x}{3} + \left(\frac{3}{n\pi} \right)^2 \cos \frac{n\pi x}{3}$$

$$\therefore a_n = \frac{2}{3} \left[\frac{3}{n\pi} x \sin \frac{n\pi x}{3} + \left(\frac{3}{n\pi} \right)^2 \cos \frac{n\pi x}{3} \Big|_0^2 + \frac{3}{n\pi} \sin \frac{n\pi x}{3} \Big|_2^3 \right]$$

$$= \frac{2}{3} \left[\frac{3}{n\pi} 2 \sin \frac{2n\pi}{3} + \left(\frac{3}{n\pi} \right)^2 \left(\cos \frac{2n\pi}{3} - 1 \right) + \frac{3}{n\pi} \left(-\sin \frac{2n\pi}{3} \right) \right]$$

$$a_n = \frac{2}{3} \left[\frac{3}{n\pi} \sin \frac{2n\pi}{3} + \left(\frac{3}{n\pi} \right)^2 \left(\cos \frac{2n\pi}{3} - 1 \right) \right]$$

$$\text{odd extension:} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3},$$

p3

$$b_n = \frac{1}{3} \int_{-3}^3 f(x) \sin \frac{n\pi x}{3} dx = \frac{2}{3} \left(\int_0^2 x \sin \frac{n\pi x}{3} dx + \int_2^3 \sin \frac{n\pi x}{3} dx \right)$$

$$\int x \sin \frac{n\pi x}{3} dx = x \left(-\frac{1}{\frac{n\pi}{3}} \cos \frac{n\pi x}{3} \right) - \int -\frac{1}{\frac{n\pi}{3}} \cos \frac{n\pi x}{3} dx$$

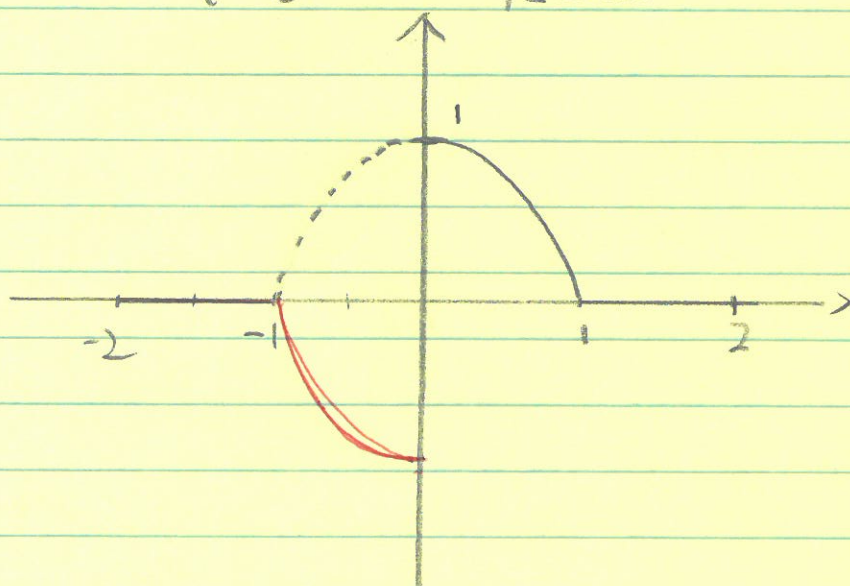
$$= -\frac{3}{n\pi} x \cos \frac{n\pi x}{3} + \left(\frac{3}{n\pi} \right)^2 \sin \frac{n\pi x}{3}$$

$$b_n = \frac{2}{3} \left(-\frac{3}{n\pi} x \cos \frac{n\pi x}{3} + \left(\frac{3}{n\pi} \right)^2 \sin \frac{n\pi x}{3} \right) \Big|_0^2 - \frac{1}{\frac{n\pi}{3}} \cos \frac{n\pi x}{3} \Big|_2^3$$

$$= \frac{2}{3} \left(-\frac{3}{n\pi} \cdot 2 \cos \frac{2n\pi}{3} + \left(\frac{3}{n\pi} \right)^2 \sin \frac{2n\pi}{3} - \frac{3}{n\pi} \left(\cos n\pi - \cos \frac{2n\pi}{3} \right) \right)$$

$$b_n = \frac{2}{3} \left(-\frac{3}{n\pi} \cos \frac{2n\pi}{3} - \frac{3}{n\pi} \cos n\pi + \left(\frac{3}{n\pi} \right)^2 \sin \frac{2n\pi}{3} \right)$$

$$(c) f(x) = \begin{cases} 1-x^2 & 0 \leq x < 1 \\ 0 & 1 \leq x < 2 \end{cases}$$



$$2L=4, \\ L=2$$

even extension

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{2}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx = \int_0^1 (1-x^2) \cos \frac{n\pi x}{2} dx$$

$$\int x^2 \cos \frac{n\pi x}{2} dx = x^2 \cdot \frac{1}{\frac{n\pi}{2}} \sin \frac{n\pi x}{2} - \int 2x \cdot \frac{1}{\frac{n\pi}{2}} \sin \frac{n\pi x}{2} dx$$

$$= \frac{2}{n\pi} x^2 \sin \frac{n\pi x}{2} - \frac{4}{n\pi} \int x \sin \frac{n\pi x}{2} dx$$

$$\int x \sin \frac{n\pi x}{2} dx = x \left(-\frac{1}{\frac{n\pi}{2}} \cos \frac{n\pi x}{2} \right) - \int -\frac{1}{\frac{n\pi}{2}} \cos \frac{n\pi x}{2} dx$$

$$= -\frac{2}{n\pi} x \cos \frac{n\pi x}{2} + \left(\frac{2}{n\pi} \right)^2 \sin \frac{n\pi x}{2}$$

P.4

$$\int x^2 \cos \frac{n\pi x}{2} dx = \frac{2}{n\pi} x^2 \sin \frac{n\pi x}{2} - \frac{4}{n\pi} \cdot \left(-\frac{2}{n\pi} x \cos \frac{n\pi x}{2} + \left(\frac{2}{n\pi} \right)^2 \sin \frac{n\pi x}{2} \right)$$

$$\therefore a_n = \left. \frac{1}{n\pi} \sin \frac{n\pi x}{2} \right|_0^1 - \left[\frac{2}{n\pi} x^2 \sin \frac{n\pi x}{2} - \frac{4}{n\pi} \left(-\frac{2}{n\pi} x \cos \frac{n\pi x}{2} + \left(\frac{2}{n\pi} \right)^2 \sin \frac{n\pi x}{2} \right) \right]_0^1$$

$$= \frac{2}{n\pi} \sin \frac{n\pi}{2} - \left[\frac{2}{n\pi} \sin \frac{n\pi}{2} - \frac{4}{n\pi} \left(-\frac{2}{n\pi} \cos \frac{n\pi}{2} + \left(\frac{2}{n\pi} \right)^2 \sin \frac{n\pi}{2} \right) \right]$$

$$a_n = -\frac{8}{(n\pi)^2} \cos \frac{n\pi}{2} + \frac{4}{n\pi} \left(\frac{2}{n\pi} \right)^2 \sin \frac{n\pi}{2}$$

odd extension

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}, \quad b_n = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx$$

$$b_n = \int_0^2 f(x) \sin \frac{n\pi x}{2} dx = \int_0^1 (1-x^2) \sin \frac{n\pi x}{2} dx = -\frac{1}{n\pi} \cos \frac{n\pi x}{2} \Big|_0^1 - \int_0^1 x^2 \sin \frac{n\pi x}{2} dx$$

$$\int x^2 \sin \frac{n\pi x}{2} dx = x^2 \left(-\frac{1}{n\pi} \right) \cos \frac{n\pi x}{2} - \int 2x \left(-\frac{1}{n\pi} \right) \cos \frac{n\pi x}{2} dx$$

$$= \left(-\frac{2}{n\pi} \right) x^2 \cos \frac{n\pi x}{2} + \frac{4}{n\pi} \int x \cos \frac{n\pi x}{2} dx$$

$$= \left(-\frac{2}{n\pi} \right) x^2 \cos \frac{n\pi x}{2} + \frac{4}{n\pi} \left(x \frac{1}{n\pi} \sin \frac{n\pi x}{2} - \int \frac{1}{n\pi} \sin \frac{n\pi x}{2} dx \right)$$

$$= \left(-\frac{2}{n\pi} \right) x^2 \cos \frac{n\pi x}{2} + \frac{4}{n\pi} \left(\frac{2}{n\pi} x \sin \frac{n\pi x}{2} + \left(\frac{1}{n\pi} \right)^2 \cos \frac{n\pi x}{2} \right)$$

$$b_n = -\frac{2}{n\pi} (\cos \frac{n\pi}{2} - 1) - \left[\left(-\frac{2}{n\pi} \right) x^2 \cos \frac{n\pi x}{2} + \frac{8}{(n\pi)^2} x \sin \frac{n\pi x}{2} + \frac{16}{(n\pi)^3} \cos \frac{n\pi x}{2} \right]_0^1$$

$$b_n = -\frac{2}{n\pi} (\cos \frac{n\pi}{2} - 1) - \left[-\frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{8}{(n\pi)^2} \sin \frac{n\pi}{2} + \frac{16}{(n\pi)^3} (\cos \frac{n\pi}{2} - 1) \right]$$