Math 222-006 (Prof. Young)

NAME: Solution

Math 222 Quiz March 7, Spring 2016
Show all your work. No calculator.

Problem 1: Consider the differential equation

$$2t^2y'' - ty' + y = t\sqrt{t}.$$

Given that $y_1 = t$ is a solution for the corresponding homogeneous equation, first find another solution y_2 for the corresponding homogeneous equation. Then find a general solution for the full non-homogeneous equation.

$y_2 = vt,\ y_2' = v't + v,\ y_2'' = v''t + 2v' \to v' = t^{-3/2},\quad v = \frac{1}{-1/2} t^{-1/2},\quad y_2 = -2t^{-1/2}\cdot t = -2t^{1/2}$

or $y_2 = t^{1/2}$

$2t^2(v''t + 2v') - t(v't + v) + vt = 0$

$2t^3v'' + 3t^2v' = 0$

$\frac{v''}{v'} = -\frac{3t^2}{2t^3} = -\frac{3}{2t}$

$\ln v' = -\frac{3}{2}\ln t = \ln t^{-3/2}$

$W(y_1, y_2) = \begin{vmatrix} t & t^{1/2} \\ 1 & \frac{1}{2}t^{-1/2} \end{vmatrix} = \frac{1}{2}t^{1/2} - t^{1/2} = -\frac{1}{2}t^{1/2}$

$Y = \left(\int \frac{-\frac{1}{2}t^{-1/2}\cdot t^{1/2}}{-\frac{1}{2}t^{1/2}}\right)t + \left(\int \frac{-\frac{1}{2}t^{-1/2}\cdot t}{-\frac{1}{2}t^{1/2}}\right)t^{1/2} = 2t^{1/2}\cdot t - t^{3/2} = t^{3/2}$

$\boxed{y = c_1 t + c_2 t^{1/2} + t^{3/2}}$

Problem 2: Find the solution of the IVP:

$$y'' + 4y = 6\sin(4t), \quad y(0) = 0, \quad y'(0) = 0.$$

$r^2 + 4 = 0,\ r = \pm 2i,$

$y_1 = \cos 2t,\ y_2 = \sin 2t$

$Y = A\cos 4t + B\sin 4t$

$Y' = -4A\sin 4t + 4B\cos 4t$

$Y'' = -16A\cos 4t - 16B\sin 4t$

$Y'' + 4Y = 6\sin 4t$

$-16A\cos 4t - 16B\sin 4t + 4(A\cos 4t + B\sin 4t) = 6\sin 4t$

$\cos 4t:\ -16A + 4A = 0,\quad \sin 4t:\ -16B + 4B = 6,\ \begin{cases} A = 0 \\ B = -1/2 \end{cases}$

$y = c_1\cos 2t + c_2\sin 2t - \frac{1}{2}\sin 4t$

$y'(t) = -2c_1\sin 2t + 2c_2\cos 2t - \frac{4}{2}\cos 4t$

$y(0) = 0 = c_1 + 0 + 0 \qquad c_1 = 0$

$y'(0) = 0 = c_2 - 2 = 0 \qquad c_2 = 2$

$y = 2\sin 2t - \frac{1}{2}\sin 4t$

Problem 3: Find the solution of the IVP:

$$y'' - 3y' - 4y = t + 2, \quad y(0) = 3, \quad y'(0) = 0.$$

What is the behavior of the solution when $t \to \infty$?

$r^2 - 3r - 4 = 0,\ (r-4)(r+1) = 0,\ r = -1, 4$

$y_1 = e^{-t}$

$y_2 = e^{4t}$

$Y = At + B$

$Y' = A$

$-3A - 4(At + B) = t + 2$

$t^1:\ -4A = 1,\ A = -1/4$

$t^0:\ -3A - 4B = 2,\ \cancel{-3A \ldots}\ B = -\frac{5}{16}$

$y = c_1e^{-t} + c_2e^{4t} - \frac{t}{4} - \frac{5}{16}$

$y' = -c_1e^{-t} + 4c_2e^{4t} - \frac{1}{4}$

$y(0) = c_1 + c_2 - \frac{5}{16} = 3,\quad c_1 + c_2 = \frac{53}{16}$

$y'(0) = -c_1 + 4c_2 - \frac{1}{4} = 0,\quad -c_1 + 4c_2 = \frac{1}{4}$

$5c_2 = \frac{57}{16},\quad c_2 = \frac{57}{80},\quad c_1 = \frac{208}{80} = \frac{13}{5}$

$y = \frac{13}{5}e^{-t} + \frac{57}{80}e^{4t} - \frac{t}{4} - \frac{5}{16},\quad y \to +\infty$ as $t \to \infty$