Math 222-006 (Prof. Young)

NAME: Solutions

Math 222 Quiz April 8, Spring 2016
Show all your work. No calculator.

Problem 1: Find the inverse Laplace transform of functions in (a) and (b). Find the Laplace transform of the function in (c). (a) $F(s) = \frac{2s-3}{s^2+2s+10}$. (b) $G(s) = \frac{(s-2)e^{-s}}{s^2-4s+3}$. (c) $f(t) = te^{at}$.

(a) $F(s) = \frac{2s-3}{(s+1)^2+3^2} = \frac{2(s+1)-6}{(s+1)^2+3^2}$, $\mathcal{L}^{-1}\left[\frac{2(s+1)-2\cdot 3}{(s+1)^2+3^2}\right] = \boxed{2\cdot e^{-t}\cos 3t - 2\cdot e^{-t}\sin 3t}$

(b) $G(s) = \frac{(s-2)e^{-s}}{(s-3)(s-1)}$, $H(s) \equiv \frac{s-2}{(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1}$, $A(s-1)+B(s-3) = s-2$

$s=1, \quad -2B=-1, \quad B=1/2$

$s=3, \quad 2A=1, \quad A=1/2$

$\mathcal{L}^{-1}[H] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-3}+\frac{1}{s-1}\right] = \frac{1}{2}(e^{3t}+e^{t}) \equiv h(t)$

$\boxed{\mathcal{L}^{-1}[e^{-s}\cdot H(s)] = u_1(t)\cdot h(t-1)}$

(c) $\boxed{\mathcal{L}[te^{at}] = \frac{1!}{(s-a)^2}}$

Problem 2: Solve the following IVP: $2y'' + y' + 2y = g(t)$, with $y(0) = 0$, $y'(0) = 0$, and

$$g(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \text{ and } t \ge 20. \end{cases}$$

$g(t) = u_5(t) - u_{20}(t)$

$\mathcal{L}[2y''+y'+2y = u_5 - u_{20}]$, $(2s^2+s+2)Y = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$, $Y = \frac{1}{s(2s^2+s+2)}(e^{-5s} - e^{-20s})$

$\mathcal{L}^{-1}\left[\frac{1}{s(2s^2+s+2)}\right] = \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{Bs+C}{2s^2+s+2}\right]$, $A(2s^2+s+2) + Bs^2+Cs = 1$

$s^2: \quad 2A+B = 0 \qquad B=-1$

$s^1: \quad 2A+C = 0 \qquad C=-1$

$s^0: \quad 2A = 1, \quad A = 1/2$

$\mathcal{L}^{-1}\left[\frac{1/2}{s} - \frac{s+1}{2(s^2+\frac{1}{2}s+\frac{1}{16})+2-\frac{1}{8}}\right]$

$= \frac{1}{2} - \mathcal{L}^{-1}\left[\frac{s+1}{2(s+\frac{1}{2})^2+\frac{15}{8}}\right] = \frac{1}{2} - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s+\frac{1}{2}+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{15}}{4})^2}\right]$

$= \frac{1}{2} - \frac{1}{2}e^{-\frac{1}{2}t}\cos\frac{\sqrt{15}}{4}t - \frac{1}{2}\cdot\frac{4}{\sqrt{15}}\cdot e^{-\frac{1}{2}t}\sin\frac{\sqrt{15}}{4}t \equiv f(t)$

$\boxed{y(t) = u_5 f(t-5) - u_{20} f(t-20)}$