HW2 Solutions, Spring 2016, MATH 222

Problem 1

a) See figure 1

b) General solution: $y' + \frac{2}{3}y = 1 - \frac{t}{2}$

so $p(t) = \frac{2}{3}$, $g(t) = 1 - t/2$, and then

$\mu(t) = e^{\int p(t)dt} = e^{2t/3}$. With this $\mu(t)$ the differential equation becomes

$$\frac{d}{dt}\Big(\underbrace{e^{2t/3}}_{\mu}\underbrace{y(t)}_{y}\Big) = \underbrace{\left(1-\frac{t}{2}\right)}_{g}\underbrace{e^{2t/3}}_{\mu} \Rightarrow$$

$$\left.\begin{aligned} e^{2t/3}y(t) &= \underbrace{\int^t \left(1-\frac{t'}{2}\right)e^{2t'/3}dt'}_{= \frac{21}{8}e^{2t/3} - \frac{3}{4}te^{2t/3}} + C_1 \end{aligned}\right\} \Rightarrow$$

$y(t) = \frac{21}{8} - \frac{3}{4}t + C_1 e^{-2t/3}$ and to satisfy the initial condition $y(0) = 2$ we need $C_1 = 2 - \frac{21}{8}$ so solution of initial value problem is $y(t) = \frac{21}{8} - \frac{3}{4}t + \left(2 - \frac{21}{8}\right)e^{-2t/3}$

c) for an arbitrary initial condition $y(0)=y_0$ the solution is $y(t) = \frac{21}{8} - \frac{3}{4}t + \left(y_0 - \frac{21}{8}\right)e^{-2t/3}$.

In this part we want to find the y_0 for which two things happen simultaneously: $y(t) = 0$ and $y'(t) = 0$ $\Rightarrow$

$$\frac{21}{8} - \frac{3t}{4} + \left(y_0 - \frac{21}{8}\right)e^{-2t/3} = 0 \quad \underline{\text{and}}$$

$$-\frac{3}{4} - \frac{2}{3}\left(y_0 - \frac{21}{8}\right)e^{-2t/3} = 0$$

Let $A = \left(y_0 - \frac{21}{8}\right)e^{-2t/3}$ so that $A = -\frac{9}{8}$ from 2nd equation. The first equation then is $\frac{21}{8} - \frac{3}{4}t - \frac{9}{8} = 0 \Rightarrow t = 2$. Then,

$A = \left(y_0 - \frac{21}{8}\right)e^{-4/3} = -\frac{9}{8}$ allows us to solve for $y_0 = -1.642876$ (see figure 2)

d) Look at the solution $y(t) = \frac{21}{8} - \frac{3}{4}t + \left(y_0 - \frac{21}{8}\right)e^{-2t/3}$.

As t gets larger and larger $\left(y_0-\frac{21}{8}\right)e^{-2t/3} \to 0$

and there is a t after which

$y(t) \approx \frac{21}{8} - \frac{3}{4}t$.

Look at figure 3. All solutions appear to become a straight line with negative slope as t increases.

Pick two pts (shown on figure 3) and compute the line eqtn as

$y = -0.7425t + 2.505$

This is the "experimentally" obtained large-t behavior of the solution which agrees with what we found above,

$y \approx \frac{21}{8} - \frac{3}{4}t$.

Problem 2

#22 Section 2.2

Solve the I.V.P. $y' = \frac{3x^2}{3y^2-4}$, $t > 1$

with initial condition $y(1) = 0$.

diff. eq. is separable $\Rightarrow$ separate variables:

$(3y^2-4)dy = 3x^2dx \Rightarrow y^3-4y = x^3+C$. Thus,

the integral curves are the equation

$y^3-4y-x^3 = C$.

To find the integral curve passing through the initial condition $y(1)=0$:

$\cancel{y^3(1)}^{\,0} - 4\cancel{y(1)}^{\,0} - 1^3 = C \Rightarrow C = -1$.

Note: The equation for the integral curve passing through the initial condition

$y^3-4y-x^3+1 = 0$ is an implicit definition

of the function $y(x)$ which satisfies

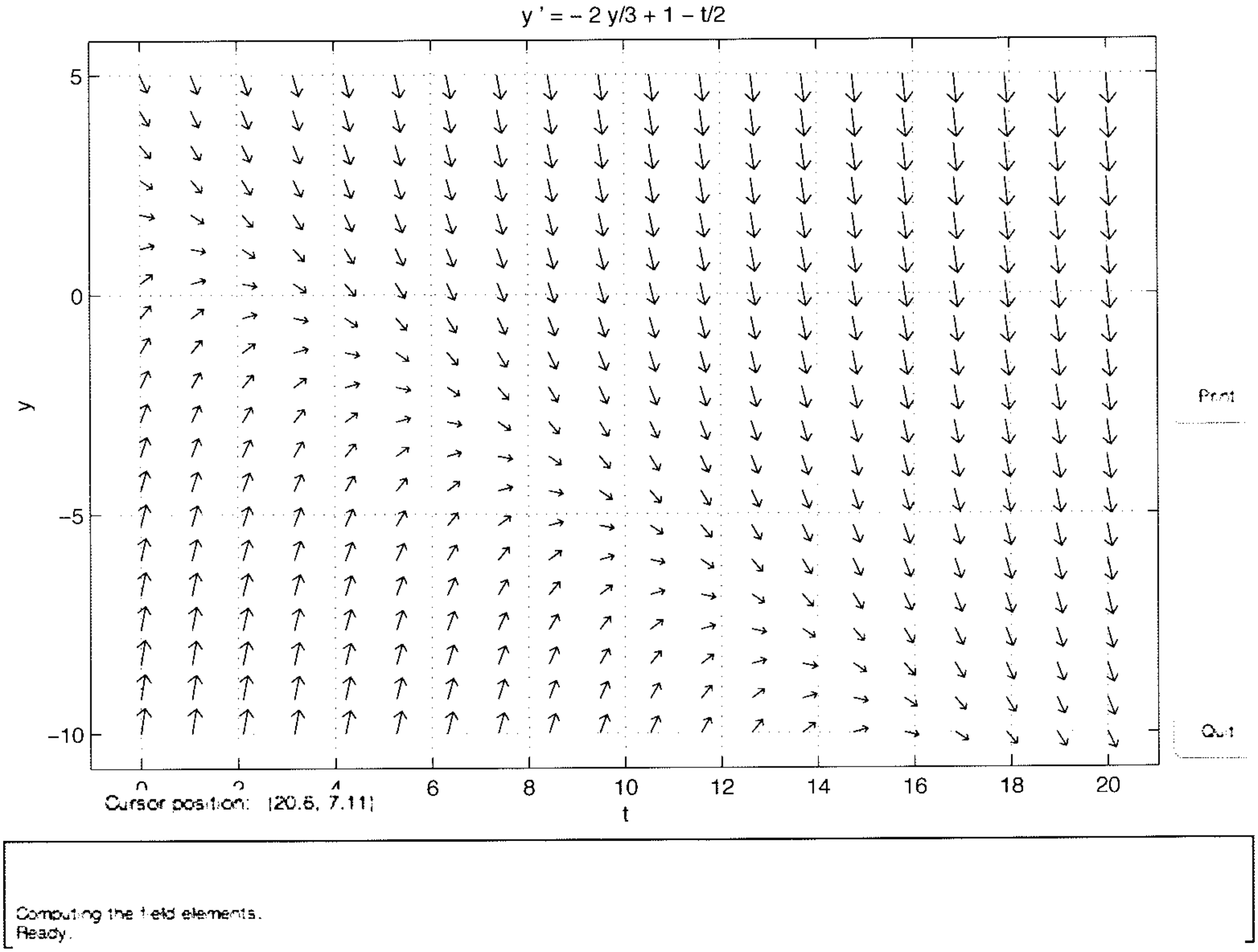
$y(1)=0$ and is the solution of the initial value problem.

The interval of validity of this solution extends on either side of the initial point as long as the function remains differentiable.

The differential equation itself, $\frac{dy}{dx} = \frac{3x^2}{3y^2-4}$, tells us that its solution, $y(x)$, will have vertical asymptotes as $y \to \pm \frac{2}{\sqrt{3}} \simeq \pm 1.1547005$

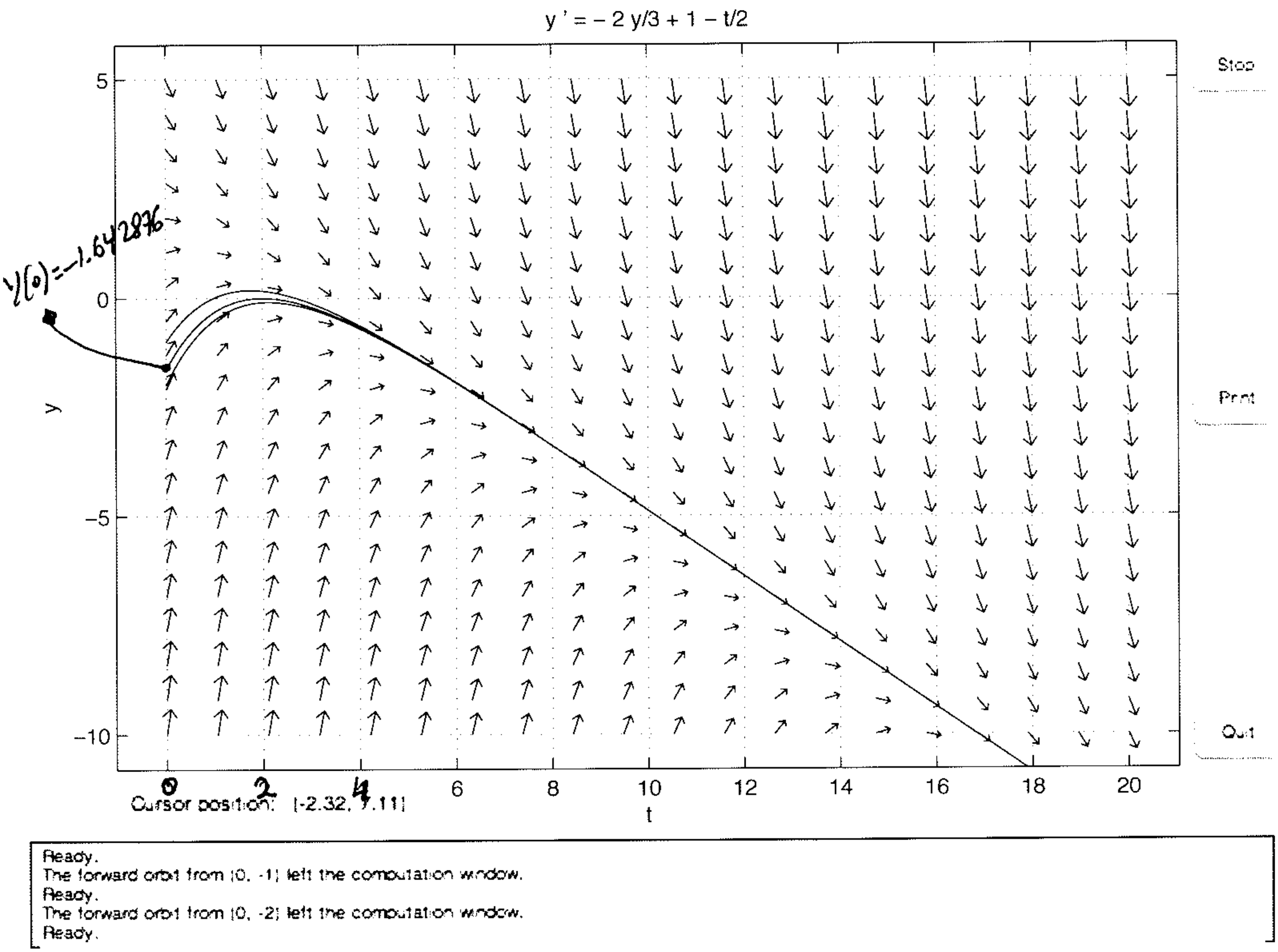
Since then $\frac{dy}{dx} \to \pm \infty$. Looking at the attached (figure 4) graph of the solution (generated with dfield) we see the interval of definition is all x between (approximately) -1.3 and 1.64.

figure 1



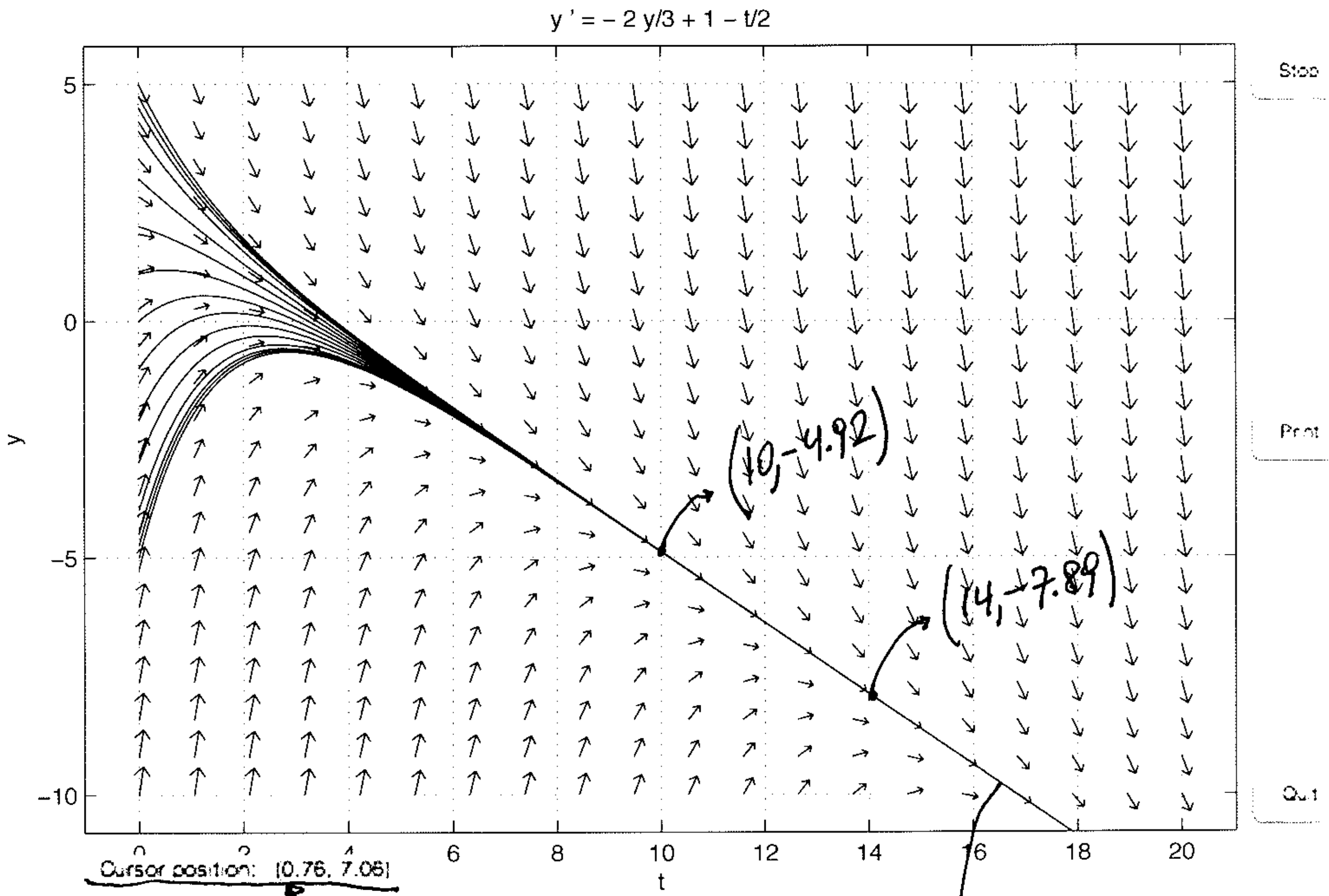
y ' = − 2 y/3 + 1 − t/2
5
0
−5
−10
y
0
2
4
6
8
10
12
14
16
18
20
t
Cursor position: [20.6, 7.11]
Print
Quit
Computing the field elements.
Ready.

figure 2



y ' = − 2 y/3 + 1 − t/2
y(0)=-1.642876
Stop
Print
Quit
y
t
Cursor position: [-2.32, 7.11]
Ready.
The forward orbit from (0, -1) left the computation window.
Ready.
The forward orbit from (0, -2) left the computation window.
Ready.

figure 3



Ready.
The forward orbit from [0, -4.8] left the computation window.
Ready.
The forward orbit from [0, -5] left the computation window.
Ready.

I picked the pts using the info on the pointer location shown here

Looks like a straight line. Pick two points on it and find the eqtn $y = at + b$

Figure 4