MATH 222 Final Exam

Dec. 18, 2015

Read each problem carefully. Show all your work for each problem. No Calculators!

1. (12 pts) Solve the following initial value problem

$$x' = \begin{pmatrix} 3 & 6 \\ -1 & 8 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

2. (13 pts) A function $f(x)$ is defined for $-1 \le x < 1$ as

$$f(x) = \begin{cases} -3 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \end{cases}, \qquad f(x+2) = f(x)$$

(a) Sketch three periods of this function. (b) Find the Fourier series of this function.

3. (10 pts) Find the eigenvalues λ and the corresponding eigenfunctions of the boundary value problem $y'' + \lambda y = 0$ with $y'(0) = 0$ and $y'(2) = 0$.

4. (14 pts) Solve the following initial value problems

$$(a)\ \frac{1}{t}\frac{dy}{dt} = \frac{2y}{1+t^2}, \quad y(0) = 1, \qquad (b)\ t\frac{dy}{dt} + 2y = e^t, \quad y(1) = 1.$$

5. (a) (8 pts) Given that $y_1(t) = t$ is the solution of the differential equation

$$t^2 y'' - t(t+2)\, y' + (t+2)y = 0,$$

find a second linearly independent solution $y_2(t)$.

(b) (8 pts) Find the general solution of $y'' + 4y' + 4y = e^{-2t}$.

6. (12 pts) Seek a power series solution of the following differential equation about $x = 0$

$$(x^2 - 1)\, y'' + xy' - y = 0.$$

(a) Find the recurrence relation.

(b) Find the first three terms in each of two linearly independent solutions y_1 and y_2. (Notice that some of the first three terms could be zero.)

7. (13 pts) Consider

$$y'' + y = \delta(t - \frac{\pi}{2}) + \alpha\, u_{\frac{\pi}{2}}(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

(a) Solve the initial value problem. (b) Find the value of α for which $y(\frac{3\pi}{2}) = 1$.

8. (10 pts) Find the particular solution of $t^2 y'' - 2y = 3t^2$ for $t > 0$.