

P. 1

Solutions of M222 Final Exam, Dec 2015.

Problem 1:

$$x' = \begin{pmatrix} 3 & 6 \\ -1 & 8 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 3-\lambda & 6 \\ -1 & 8-\lambda \end{pmatrix} = 0 \quad (3-\lambda)(8-\lambda) + 6 = 0, \quad \lambda^2 - 11\lambda + 24 + 6 = 0$$

$$\lambda^2 - 11\lambda + 30 = 0, \quad (\lambda - 6)(\lambda - 5) = 0, \quad \lambda = 5, 6$$

$$\lambda = 5, \quad \begin{pmatrix} 3-5 & 6 \\ -1 & 8-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$-2x_1 + 6x_2 = 0$$

$$\frac{x_1}{x_2} = \frac{6}{2} = 3, \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ is an eigen-vector}$$

$$\lambda = 6, \quad \begin{pmatrix} 3-6 & 6 \\ -1 & 8-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$-3x_1 + 6x_2 = 0,$$

$$\frac{x_1}{x_2} = \frac{6}{3} = 2, \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ is an eigen-vector}$$

$$x = c_1 e^{5t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

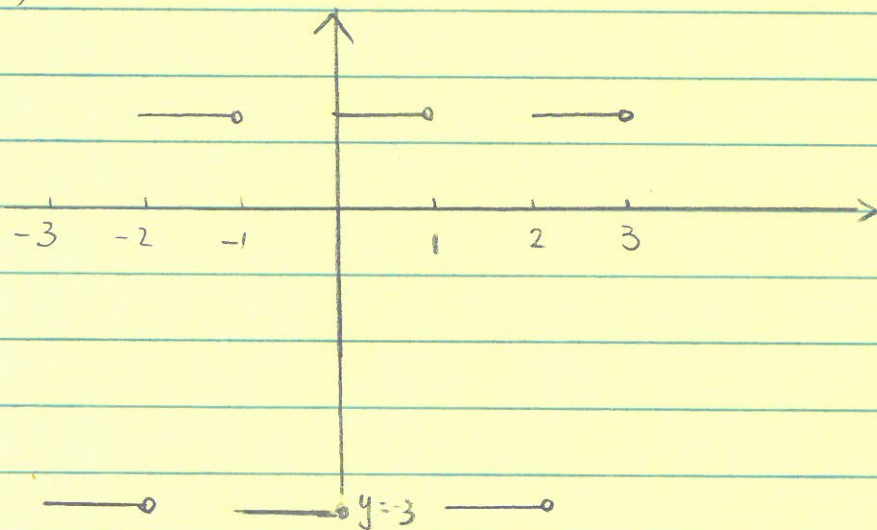
$$x(0) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\begin{aligned} 3c_1 + 2c_2 &= 5 \\ c_1 + c_2 &= 1 \end{aligned}$$

$$\boxed{\begin{aligned} c_1 &= 3 \\ c_2 &= -2 \end{aligned}}$$

$$\boxed{x(t) = 3e^{5t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2e^{6t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

Problem 2: (a)



$$(b) \quad f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} + \frac{a_0}{2}$$

$$\int_{-1}^1 \frac{a_0}{2} dx = \int_{-1}^1 f(x) dx = -3 + 1 = -2 = \frac{a_0}{2} \cdot 2 \quad \boxed{a_0 = -2}$$

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$$\int_{-1}^1 f(x) \cos \frac{n\pi x}{L} dx = \int_{-1}^1 a_n \cdot \cos^2 \frac{n\pi x}{L} dx = \int_{-1}^1 \frac{a_n (1 + \cos \frac{2n\pi x}{L})}{2} dx$$

$$\int_{-1}^0 -3 \cdot \cos \frac{n\pi x}{L} + \int_0^1 1 \cdot \cos \frac{n\pi x}{L} dx = a_n, \quad n \neq 1$$

$$-3 \frac{1}{\frac{n\pi}{L}} \sin \frac{n\pi x}{L} + \left. \frac{1}{\frac{n\pi}{L}} \sin \frac{n\pi x}{L} \right|_0 = 0 = a_n, \quad n \geq 1$$

$$\int_{-1}^1 f(x) \sin \frac{n\pi x}{L} dx = \int_{-1}^1 b_n \sin^2 \frac{n\pi x}{L} dx = \int_{-1}^1 \frac{b_n}{2} (1 - \cos 2\frac{n\pi x}{L}) dx = b_n$$

$$\int_{-1}^0 -3 \sin \frac{\pi x}{2} dx + \int_0^1 1 \sin \frac{\pi x}{2} dx = b_n$$

$$\frac{3}{\frac{n\pi}{L}} \cos \frac{n\pi x}{L} \Big|_{-1}^0 - \frac{1}{\frac{n\pi}{L}} \cos \frac{n\pi x}{L} \Big|_0^1 = \frac{3}{\frac{n\pi}{L}} (1 - (-1)^n) - \frac{1}{\frac{n\pi}{L}} ((-1)^n - 1)$$

$$= \frac{4L}{n\pi} - \frac{4L}{n\pi} (-1)^n = \frac{8}{n\pi} (1 - (-1)^n)$$

Problem 3: $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(2) = 0$

$$\lambda \neq 0, \quad y = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x \quad \lambda = 0 \quad y = ax + b$$

$$y' = \sqrt{-\lambda} (-A \sin \sqrt{-\lambda} x + B \cos \sqrt{-\lambda} x) \quad y' = a = 0$$

$$y = b$$

$$y'(0) = 0 = \sqrt{\lambda} \cdot B \cdot \cos \sqrt{\lambda} \cdot 0 \Rightarrow B = 0$$

$$y'(z) = 0 = \sqrt{-\lambda} A \cos \sqrt{-\lambda} z \quad 2\sqrt{-\lambda} = \frac{\pi}{2} + n\pi, \quad \sqrt{-\lambda} = \frac{\pi}{4} + \frac{n\pi}{2}$$

$$-\lambda = \left(\frac{\pi}{4} + n\pi\right)^2, \quad \lambda = -\left(\frac{\pi}{2} + \frac{1}{4}\right)^2 \pi^2$$

$$\lambda > 0 \quad y=0 \Rightarrow \text{not an eigenvalue}$$

$$\Rightarrow \lambda = 0, y = 1$$

$$\lambda = -\left(\frac{n}{2} + \frac{1}{4}\right)^2 \pi^2 < 0, \quad y = \left(\frac{n}{2} + \frac{1}{4}\right) \pi \sin \left(\frac{n}{2} + \frac{1}{4}\right) \pi x$$

Problem 4: (a) $\frac{1 dy}{t \sqrt{t}} = \frac{2y}{1+t^2}, y(0) = 1$

$$\frac{dy}{y} = \frac{2t}{1+t^2} dt,$$

$$\ln y = \ln(1+t^2) + C$$

$$\ln 1 = 0 = \ln 1 + C, \quad C = 0$$

$$\rightarrow y = (1+t^2)$$

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(b) $t \frac{dy}{dt} + 2y = e^t, \quad y(1) = 1$

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{e^t}{t}, \quad \mu = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = e^{\ln t^2} = t^2$$

$$t^2 y' + 2ty = (t^2 y)' = t e^t, \quad \int (t^2 y)' dt = \int t e^t dt + C$$

$$t^2 y = t e^t - e^t + C, \quad t=1, y(1)=1, \quad 1 \cdot 1 = 1 e^1 - e^1 + C, C=1$$

$$y = \frac{1}{t} e^t - \frac{1}{t^2} e^t + \frac{1}{t^2}$$

Problem 5: (a) $y_2(t) = y_1 \cdot v(t) = t v$

$$y_2' = v + t v', \quad y_2'' = v' + v' + t v'' = 2v' + t v''$$

$$t^2 (2v' + t v'') - t(t+2)(v + t v') + (t+2)t v = 0$$

$$t^3 v'' + 2t^2 v' - t^2(t+2)v' - t(t+2)v + t(t+2)v = 0$$

$$t^3 v'' - t^3 v' = 0, \quad v'' - v' = 0, \quad v' = v', \quad \ln v' = t, \quad v' = e^t, \quad v = e^t$$

$$y_2 = t e^t$$

(b) $y'' + 4y' + 4y = e^{-2t}$

$$r^2 + 4r + 4 = 0, \quad r = -2, -2$$

$$y_1 = e^{-2t}, \quad y_2 = t e^{-2t}, \quad Y = A t^2 e^{-2t}$$

$$Y' = A 2t e^{-2t} - A 2t^2 e^{-2t}$$

$$Y'' = 2A e^{-2t} - 4A t e^{-2t} - 4A t e^{-2t} + 4A t^2 e^{-2t}$$

$$(2A - 8At + 4At^2) e^{-2t} + A(8t - 8t^2) e^{-2t} + 4A t^2 e^{-2t} = e^{-2t}$$

$$2A e^{-2t} = e^{-2t}, \quad A = \frac{1}{2}$$

general solution $y = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{2} t^2 e^{-2t}$

Problem 6:

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(x^2 - 1) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x^0: -2 \cdot 1 \cdot a_2 - a_0 = 0, \quad a_2 = -\frac{1}{2} a_0$$

$$x^1: -3 \cdot 2 \cdot a_3 + a_1 - a_1 = 0, \quad a_3 = 0$$

$$x^2: 2 \cdot 1 \cdot a_2 - 4 \cdot 3 \cdot a_4 + 2a_2 - a_2 = 0, \quad -12a_4 + 3a_2 = 0, \quad a_4 = \frac{1}{4} a_2$$

$$x^3: 3 \cdot 2 \cdot a_3 - 5 \cdot 4 \cdot a_5 + 3a_3 - a_3 = 0, \quad -20a_5 = -8a_3, \quad a_5 = \frac{2}{5} a_3 = 0$$

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$$x^n: \quad n(n-1)a_n - (n+2)(n+1)a_{n+2} + n a_n - a_n = 0,$$

$$(n+1)(n-1)a_n = (n+1)(n+2)a_{n+2}, \quad a_{n+2} = \frac{n-1}{n+2} a_n,$$

$$y_1 = a_0 \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots \right)$$

$$y_2 = a_1 (x + 0x^3 + 0x^5 + \dots)$$

Problem 7:

$$y'' + y = \delta(t - \frac{\pi}{2}) + \alpha U_{\frac{\pi}{2}}(t), \quad y(0) = 0, \quad y'(0) = 1$$

$$\mathcal{L}[y'' + y] = \mathcal{L}[\delta(t - \frac{\pi}{2})] + \alpha \mathcal{L}[U_{\frac{\pi}{2}}(t)] = e^{-\frac{\pi}{2}s} + \alpha \frac{e^{-\frac{\pi}{2}s}}{s}$$

$$s^2 Y - s y(0) - y'(0) + Y = e^{-\frac{\pi}{2}s} \left(1 + \frac{\alpha}{s} \right),$$

$$(s^2 + 1)Y = 1 + e^{-\frac{\pi}{2}s} \left(1 + \frac{\alpha}{s} \right),$$

$$Y = \frac{1}{s^2 + 1} + e^{-\frac{\pi}{2}s} \left(\frac{1}{s^2 + 1} + \alpha \frac{1}{s(s^2 + 1)} \right)$$

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}$$

$$A(s^2 + 1) + Bs^2 + Cs = 1$$

$$A + B = 0$$

$$C = 0$$

$$A = 1$$

$$B = -1$$

$$C = 0$$

$$A = 1$$

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}$$

$$y(t) = \mathcal{L}^{-1} \left[\frac{1}{s^2 + 1} + e^{-\frac{\pi}{2}s} \left(\frac{1}{s^2 + 1} \right) + \alpha e^{-\frac{\pi}{2}s} \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right) \right]$$

$$y(t) = \sin t + U_{\frac{\pi}{2}} \sin(t - \frac{\pi}{2}) + \alpha U_{\frac{\pi}{2}} (1 - \cos(t - \frac{\pi}{2}))$$

$$y(\frac{3\pi}{2}) = 1 = \sin \frac{3\pi}{2} + \sin \pi + \alpha (1 - \cos \pi)$$

$$= -1 - 2\alpha = 1, \quad -2\alpha = 2, \quad \alpha = -1$$

Problem 8:

$$t^2 y'' - 2y = 3t^2, \quad t > 0$$

$$r(r-1) - 2 = 0, \quad r^2 - r - 2 = 0, \quad (r-2)(r+1) = 0, \quad r = -1, 2$$

$$y_1 = t^{-1}$$

$$y_2 = t^2$$

$$Y = \left(\int \frac{-t^2 \cdot 3}{W(y_1, y_2)} dt \right) t^{-1} + \left(\int \frac{t^{-1} \cdot 3}{W(y_1, y_2)} dt \right) t^2$$

$$W(y_1, y_2) = \begin{vmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{vmatrix}$$

$$= \left(\int \frac{-3t^2}{2t+1} dt \right) t^{-1} + \left(\int \frac{3t^{-1}}{2t+1} dt \right) t^2$$

$$= 2t + 1$$

$$= -3 \left(\frac{1}{2} t \cdot \frac{1}{4} + \frac{t}{2t+1} \right) t^{-1} + 3 \left(\frac{1}{t} + \frac{-2}{2t+1} \right) t^2$$

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$$Y = -3 \left[\frac{1}{4} t^2 - \frac{1}{4} t \right] + \frac{1}{8} \ln(2t+1) \Big] t^1 + 3 \left[\ln t - \ln(2t+1) \right] t^2$$