# Math 222 Spring 2016, Additional Engineering Applications of Differential Equations

## 1 Leaking Bucket

This engineering application may be used to augment the materials in §2.3: Modeling with First Order Equations. This application is problem # 6 in §2.3.

The leaking bucket in figure 1 can be described by investigating the water level $h(t)$ as a function of time. The volume conservation of water in the system is represented by the

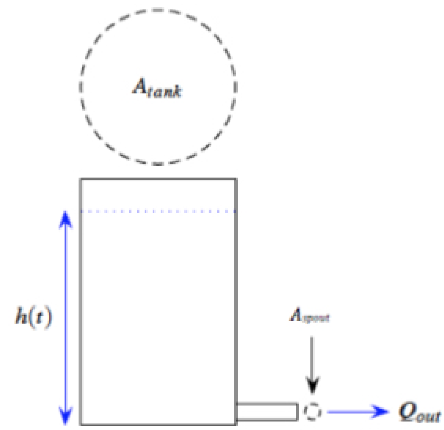



Figure 1: Sketch of a leaking bucket.

balance of volumetric flow rate $Q$ as follows:

$$Q_{in} - Q_{out} = Q_{stored}. \tag{1}$$

In the case when no water is flowing into the tank, $Q_{in} = 0$, we obtain

$$Q_{stored} = -Q_{out}. \tag{2}$$

The volumetric flow rate $Q_{stored}$ can be calculated by multiplying the velocity by the area of the tank

$$Q_{stored} = A_{tank}\frac{dh(t)}{dt}. \tag{3}$$

$Q_{out}$ is computed by multiplying the flow velocity by the area of the spout $A_{spout}$

$$Q_{out} = A_{spout}v(t), \tag{4}$$

where $v(t)$ is the velocity of water coming out of the straw. For fluids of height $h(t)$, the velocity of water coming out at the bottom is $v(t) = \sqrt{2gh(t)}$. Therefore we arrive at the governing equation for $h(t)$ as

$$A_{tank}\frac{dh(t)}{dt} = -A_{spout}\sqrt{2gh}. \tag{5}$$

Rearranging terms, we obtain the following equation

$$\frac{dh(t)}{dt} = -K\sqrt{h}, \tag{6}$$

with $K = \frac{A_{spout}}{A_{tank}}\sqrt{2g} > 0$. Before solving equation 6, we observe that the water height is decreasing with time as $\frac{dh}{dt} < 0$ for all $h \geq 0$. With the initial condition $h(0) = h_0$, equation 6 can be solved by separation of variables as follows.

$$\frac{dh}{dt} = -K\sqrt{h}, \quad \rightarrow \quad \frac{dh}{\sqrt{h}} = -Kdt. \tag{7}$$

Integrating both sides

$$\int \frac{dh}{\sqrt{h}} = \int -Kdt, \quad \rightarrow \quad 2\sqrt{h} = -Kt + c, \tag{8}$$

where the integration constant $c = 2\sqrt{h_0}$. Thus the water height can be expressed explicitly in terms of time as

$$h(t) = \left(\sqrt{h_0} - \frac{Kt}{2}\right)^2. \tag{9}$$

Note that the solution $h(t)$ in equation 9 decreases from the initial height $h_0$, and at time $t_{end} = \frac{2\sqrt{h_0}}{K}$, the water is completely drained out (by gravity) and $h(t_{end}) = 0$.

# 2 Forced Vibrations

## 3.8 Forced Vibrations

1) WITH DAMPING $\quad mu'' + \gamma u' + ku = F_0 \cos\omega t \quad (8)$ $\quad \gamma > 0$ damping

$F_0 \cos\omega t$ 'harmonic' forcing

Solution $u(t) = u_c(t) + u_p(t)$

complementary solution $Lu_c = 0$ $\quad$ particular solution $Lu_p = F_0\cos\omega t$

From § 3.7

$$u_c(t) = \begin{cases} Ae^{r_1 t} + Be^{r_2 t} & \gamma^2 - 4km > 0, \quad r_1, r_2 < 0 \\ (A+Bt)e^{-\frac{\gamma t}{2m}} & \gamma^2 - 4km = 0 \quad r_1 = r_2 = -\frac{\gamma}{2m} \\ Re^{-\frac{\gamma t}{2m}}\cos(\mu t - \delta) & \gamma^2 - 4km < 0 \quad \mu = \frac{\sqrt{(4km-\gamma^2)}}{2m} \end{cases}$$

with damping, $\gamma > 0$, $u_c(t) \to 0$ as $t \to \infty$. So, $u_c(t)$, which contains the initial condition data $u(0)$ and $u'(0)$ in $A, B, R$, and $\delta$, is 'TRANSIENT' - negligible for large times.

$$u_p(t) = A\cos\omega t + B\sin\omega t \qquad \text{periodic, steady-state (forced)}$$
$$= R\cos(\omega t - \delta) \quad (10) \qquad \text{large-time response.}$$

Find $R, \delta$ by substitution in ODE. $\left(u_p' = -R\omega\sin(\omega t - \delta),\ u_p'' = -R\omega^2\cos(\omega t-\delta)\right)$

$$R\left(-m\omega^2\cos(\omega t - \delta) - \gamma\omega\sin(\omega t-\delta) + k\cos(\omega t - \delta)\right) = F_0\cos\omega t$$

USE $\cos(A-B) = \ldots$, $\sin(A-B) = \ldots$

$$\Rightarrow \frac{R}{F_0}\Big(\left\{(k-m\omega^2)\cos\delta + \gamma\omega\sin\delta\right\}\cos\omega t$$
$$+ \left\{(k-m\omega^2)\sin\delta - \gamma\omega\cos\delta\right\}\sin\omega t\Big) = \cos\omega t$$

equate (lin. indep) $\sin\omega t$, $\cos\omega t$ terms, Put $\omega_0^2 = \frac{k}{m}$ to eliminate $k$.

$\sin\omega t$: $\quad m(\omega_0^2-\omega^2)\sin\delta - f\omega\cos\delta = 0 \qquad$ (i)

$\cos\omega t$: $\quad f\omega\sin\delta + m(\omega_0^2-\omega^2)\cos\delta = \dfrac{F_0}{A} \qquad$ (ii)

Solve $m(\omega_0^2-\omega^2)$(i) + $f\omega$(ii) $\Rightarrow$.

$$\left(m^2(\omega_0^2-\omega^2)^2+f^2\omega^2\right)\sin\delta = \frac{F_0 f\omega}{A}.$$

$m(\omega_0^2-\omega^2)$(ii) $-$ $f\omega$(i) $\Rightarrow$.

$$\left(m^2(\omega_0^2-\omega^2)^2+f^2\omega^2\right)\cos\delta = \frac{F_0 m}{A}\left(\omega_0^2-\omega^2\right)$$

Put $\Delta = \sqrt{\left(m^2(\omega_0^2-\omega^2)^2+f^2\omega^2\right)}$. Square and add $\Rightarrow$.

$$\Delta^4\underbrace{\left(\cos^2\delta+\sin^2\delta\right)}_{=1} = \left(\frac{F_0}{A}\right)^2\cdot\Delta^2$$

$$\Rightarrow \quad A = \frac{F_0}{\Delta}, \quad \text{then } \cos\delta = \frac{m(\omega_0^2-\omega^2)}{\Delta}, \quad \sin\delta = \frac{f\omega}{\Delta} \qquad (11)$$

$$x_p = A\cos(\omega t - \delta)$$

AMPLITUDE. PHASE DIFFERENCE, BETWEEN RESPONSE $x_p$ AND FORCING $F_0\cos\omega t$

How do $A$ and $\delta$ depend on $\omega$? $\qquad \Delta = \sqrt{\left(m^2(\omega_0^2-\omega^2)^2+f^2\omega^2\right)}$, $\quad \omega_0^2 = k/m \qquad$ (12)

(11), (12) $\Rightarrow$.

(i) $\underline{\underline{A}}$

$$\frac{Ak}{F_0} = \frac{1}{\sqrt{\left(\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\Gamma\frac{\omega^2}{\omega_0^2}\right)}} \qquad (13) \qquad \left(\frac{Ak}{F_0} = \frac{k}{\Delta}\right)$$

where $\Gamma = \dfrac{f^2}{mk}$.

Here, each 'group': $\dfrac{Ak}{F_0}$, $\dfrac{f^2}{mk}$, $\dfrac{\omega}{\omega_0}$ IS DIMENSIONLESS. — WE HAVE GONE FROM 5 DIMENSIONAL PARAMETERS ($m, f, k, F_0, \omega$) TO 3 DIMENSIONLESS GROUPS.

$\omega \to 0$; $\dfrac{Ak}{F_0} \to 1$. $\quad \omega \to \infty$; $\dfrac{Ak}{F_0} \sim \dfrac{\omega_0^2}{\omega^2} \to 0$

PER 'STATIC' ELONGATION OF SPRING

LIMIT OF HIGH FREQ. FORCING — NO AMPLITUDE

At what forcing frequency $\omega$ is the amplitude a maximum?

$$`Q' = \left(\frac{Rk}{F_0}\right)^2 = \frac{1}{(1-\rho)^2 + \Gamma\rho} \quad \text{where } \rho \equiv \frac{\omega^2}{\omega_0^2}. \quad \frac{d}{d\rho}\left((1-\rho)^2 + \Gamma\rho\right) = 0 \Rightarrow$$

$$-2(1-\rho) + \Gamma = 0 \Rightarrow \rho = \frac{\omega^2}{\omega_0^2} = 1 - \frac{\Gamma}{2} \Rightarrow \omega_m^2 = \omega_0^2\left(1 - \frac{\Gamma}{2}\right) \left(\overset{SO}{<} \omega_0^2\right)$$

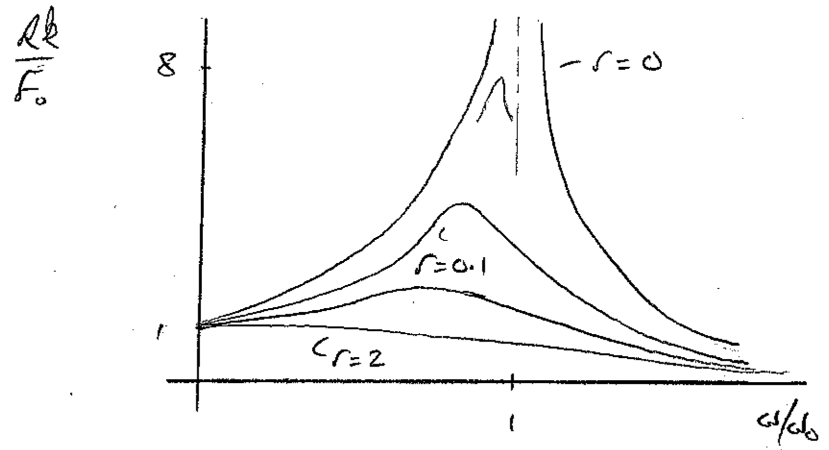
$$\underline{\text{or}} \quad \omega_m^2 = \omega_0^2\left(1 - \frac{f^2}{2mk}\right)$$

FREQ. THAT CAUSES MAX. AMPLITUDE.

$$\text{Then } `Q' = \frac{1}{\Gamma(1-\frac{\Gamma}{4})} \Rightarrow \left.\frac{Rk}{F_0}\right|_m = \frac{k}{f\omega_0\left(1 - \frac{f^2}{4mk}\right)^{\frac{1}{2}}}.$$

MAX AMPLITUDE

For small damping, $f \ll \sqrt{(mk)} = \frac{k}{\omega_0}$. $\left.\frac{Rk}{F_0}\right|_m \simeq \frac{k}{f\omega_0} \to \infty$ as $f \to 0$.




FOR $\Gamma \simeq .015$
MAX AMP. $\frac{Rk}{F_0} \simeq 8$
"NEAR-RESONANCE"
(This is fig 3.82, p 211 in book)

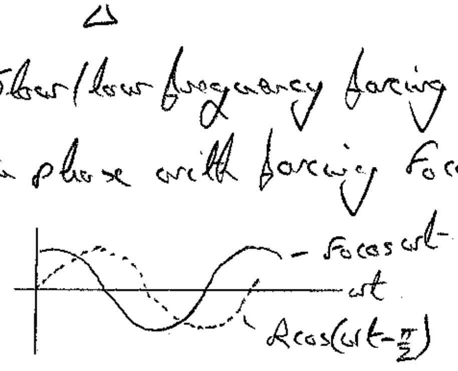
ZERO DAMPING: $f = 0 \Rightarrow \Gamma = 0$. Then (13) $\Rightarrow$. $\frac{Rk}{F_0} = \frac{1}{\left|1 - \frac{\omega^2}{\omega_0^2}\right|}$ $\nearrow \infty$ as $\omega \to \omega_0$

(ii) $\delta$ - PHASE SHIFT  From (11) $\sin\delta = \frac{f\omega}{\Delta}$, $\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}$, $\Delta = \sqrt{(m^2(\omega_0^2 - \omega^2)^2 + f^2\omega^2)}$

$\omega \to 0$: $\sin\delta \to 0$, $\cos\delta \to 1 \Rightarrow \delta \to 0$. Slow/low frequency forcing $\Rightarrow$ response $u_p = R\cos(\omega t - \delta)$ nearly in phase with forcing $F_0\cos\omega t$.

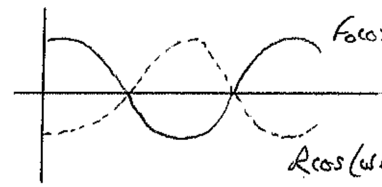
$\frac{\omega}{\omega_0} = 1$; $\sin\delta = 1$, $\cos\delta = 0 \Rightarrow \delta = \frac{\pi}{2}$.

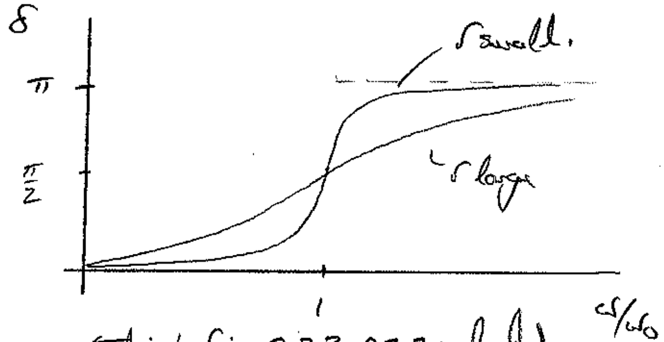
RESPONCE LAGS BEHIND FORCING BY $\frac{\pi}{2}$



$\frac{\omega}{\omega_0} \gg 1$: $\left(\frac{\omega}{\omega_0} \to \infty\right)$ $\sin\delta \simeq \frac{\gamma}{m\omega} \to 0$, $\cos\delta \to -1 \Rightarrow \delta \to \pi$.

RESPONSE LAGS FORCING BY $\pi$
"$\pi$ OUT OF PHASE".




(This is Fig. 3.8.3, p 212 in book)

small damping
$\left(\Gamma = \frac{\gamma^2}{mk} \ll 1\right)$
rapid change in $\delta$
from 0 to $\pi$ for $\omega \simeq \omega_0$

2) FORCED, WITH NO DAMPING. (p 214)

NOT RESONANT.

$$mu'' + ku = F_0 \cos\omega t. \qquad (17) \qquad \omega \neq \omega_0 \quad \left(\omega_0^2 = k/m\right)$$

General solution

$$u(t) = \underbrace{c_1\cos\omega_0 t + c_2 \sin\omega_0 t}_{u_c} + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos\omega t \qquad (18)$$

$u_c$, solution with no forcing PERIODIC - not transient

$u_p$ CHECK THIS. BY SUBSTITUTION OR UNDETERMINED COEFFICIENTS.

$c_1$ and $c_2$ determined by IC's. Take example of motion beginning from zero displacement ($u(0)=0$) at rest ($u'(0)=0$) - only the forcing excites. $u(0)=0 \Rightarrow$

$c_1 = -\frac{F_0}{m(\omega_0^2-\omega^2)}$, $u'(0)=0 \Rightarrow c_2 = 0$. Then

$$u = \frac{F_0}{m(\omega_0^2-\omega^2)}\left(\cos\omega t - \cos\omega_0 t\right) \qquad (20)$$

EQUAL MAGNITUDES

Recall $\cos(A-B) - \cos(A+B) = 2\sin A \cdot \sin B$.

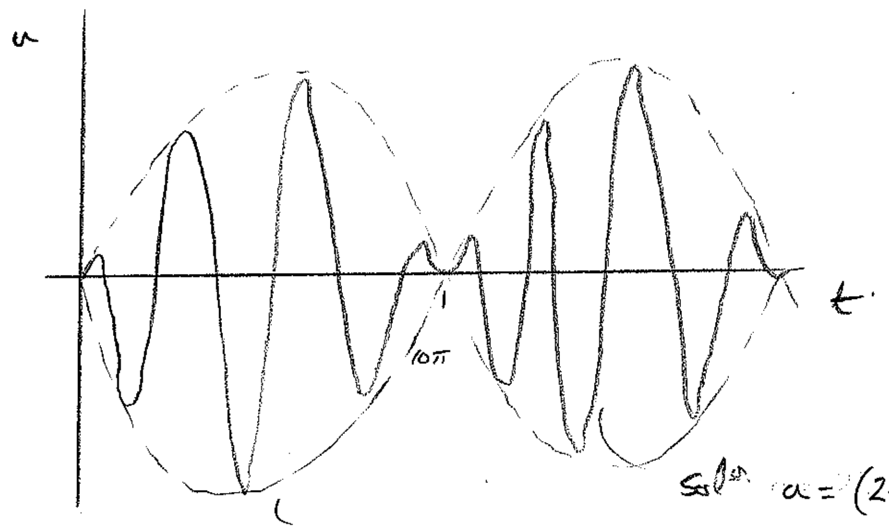
Put $\left.\begin{array}{l} A-B = \omega t \\ A+B = \omega_0 t \end{array}\right\} \Rightarrow A = \left(\frac{\omega_0+\omega}{2}\right)t,\ B = \left(\frac{\omega_0-\omega}{2}\right)t$.

$$\Rightarrow \quad u = \frac{2f_0}{m(\omega_0^2-\omega^2)} \cdot \sin\frac{(\omega_0-\omega)t}{2} \cdot \sin\frac{(\omega_0+\omega)t}{2}. \qquad (21)$$

For $\omega$ CLOSE TO $\omega_0$
BUT $\omega \neq \omega_0$.
NEAR RESONANCE
NO DAMPING.

$\omega_0-\omega$ SMALL
low frequency
'amplitude modulation'
BEATS

$\frac{\omega_0+\omega}{2} \simeq \omega_0 t$
'natural freq $\omega_0 \simeq \omega$.




BOOK EXAMPLE..
$\omega_0=1$
$\omega=0.8$.
(Fig 3.8.7, p215)

Sol$^n$ $u=(2.77\ldots)\sin\frac{t}{10}\sin\frac{9t}{10}$

ENVELOPE
$\simeq -2.7\sin\frac{t}{10}$

3) FORCED, NO DAMPING
RESONANT (p215)

$mu''+ku = f_0\cos\omega_0 t$. $\qquad \omega_0^2=\frac{k}{m}$.

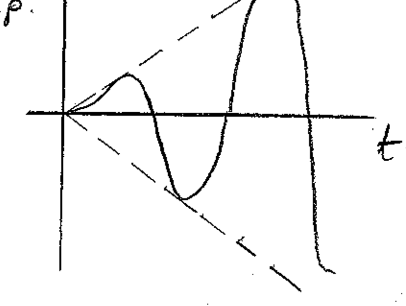
$\omega=\omega_0$ 'Full' ansatz for $u_p = (At+B)\cos\omega_0 t + (Ct+D)\sin\omega_0 t$

ZERO, NOT NEEDED
HOMOGENEOUS SOLN.
$C=\frac{f_0}{2m\omega_0}$

The general solution is

$$u = \underbrace{c_1\cos\omega_0 t + c_2\sin\omega_0 t}_{u_c \text{ ICs GIVE } c_1,c_2} + \frac{f_0}{2m\omega_0}\, t\sin\omega_0 t.$$

$u_p$.




$u_p$ OSCILLATORY AND 'GROWS WITHOUT BOUND' FOR LARGE $t$. (linear / small-amplitude model 'FAILS' as $u\uparrow$)