Math 222 Exam 1, February 17, 2016

Read each problem carefully. Show all your work. Turn off your phones. No calculators!

1. Consider the differential equation $y' = y(y^2 - 4)$.

   (a) (3 points) Find the equilibrium solution(s) ($y' = 0$).

   (b) (3 points) Sketch the direction field. (4 points) Sketch the solution $y(t)$ with initial condition $y(0) = 1$. Sketch another solution $y(t)$ with initial condition $y(1) = 3$.

2. (a) (4 points) Determine the order of the differential equation and state whether the equation is linear/nonlinear:

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{\ln x}{xy}.$$

   (b) (6 points) Verify that each given function (y_1 and y_2) is a solution of the differential equation: $2t^2y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}$.

3. Consider the IVP: $t(t-4)y'' - 3ty' + 4y = 2$, $y(3) = 0$ and $y'(3) = -1$.

   (a) (5 points) Determine the longest interval(s) in which the IVP is certain to have a unique twice-differential solution.

   (b) (5 points) Compute the Wronskian for the ODE using Abel's Theorem.

4. (a) (7 points) Find the solution of the IVP: $\frac{dy}{dx} = \frac{x}{y(1+x^2)}, \quad y(0) = -2$.

   (b) (8 points) Find the solution of the IVP: $y' + y = e^{-t}, \quad y(0) = y_0$. Find the value of y_0 such that the solution $y(t)$ reaches maximum at $t = 4$.

5. A tank with a capacity of 6 L initially contains 10 g of salt and 1 L of water. A solution containing 1 g/L of salt enters the tank at a rate of 3 L/hour. The tank is equipped with a valve so the well-stirred mixture leaves the tank at a rate $0.5V(t)$ L/hour, where $V(t)$ is the volume of solution in the tank.

   (10 points) First formulate the IVP for the volume of solution in the tank. Solve for $V(t)$. (5 points) Then show that the differential equation for the total amount of salt $Q(t)$ is the same as the differential equation for the total volume $V(t)$.

6. (10 points) Consider the IVP: $y' = -y + 1 - t$, $y(0) = 1$. Use Euler's method with a step size $h = 0.1$ to find an approximate value of the solution at $t = 0.2$.

7. (a) (5 points) Find a linear homogeneous differential equation with constant coefficients that would have $r_1 = -2$ and $r_2 = 3$ for the roots of its characteristic equation.

   (b) (10 points) Find α and β so that the IVP $y'' - y' - 2y = 0$, $y(0) = \alpha$ and $y'(0) = \beta$ would have a solution that remains bounded as $t \to \infty$.

   (c) (15 points) Solve the IVP: $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$. Sketch the solution $y(t)$.