Math 222 Exam 2, March 9, 2016

Read each problem carefully. Show all your work. Turn off your phones. No notes, books, and calculators.

1. (15 points) Find α such that the solution $y(t)$ to the IVP

$$4y'' + 12y' + 9y = 0, \quad y(0) = -1, \quad y'(0) = \alpha$$

changes sign (from negative to positive or from positive to negative) at $t = 0.5$. How many times can this solution change sign?

2. (a) (10 points) Consider the ODE: $t^2y'' - 3ty' + 4y = 0$. Given that $y_1 = t^2$ is a solution of the ODE, use reduction of order to find another fundamental solution y_2.

(b) (5 points) Use the method of undetermined coefficients to determine a suitable form (WITHOUT solving for the coefficients) for the particular solution y_p of the ODE

$$y'' + 5y' + 6y = -t + e^{-3t} + te^{-2t} + e^{-3t}\cos(t).$$

3. (15 points) First find the general solution to the ODE for $y(t)$

$$y'' + 2y' + 2y = \cos(t).$$

Then predict the behavior of the solution as $t \to \infty$.

4. (15 points) Solve the IVP: $y'' - 2y' + y = t^{-2}e^t$, $y(1) = e$, $y'(1) = 0$. Describe the behavior of $y(t)$ as $t \to \infty$.

5. A mass weighing 0.5 lb (of mass 1/64 lb·s^2/ft) stretches a spring 0.5 ft.

(a) (12 points) If there is no damping, and at $t = 0$ the mass is pulled down an additional 0.5 ft and then released from rest, determine the displacement u of the mass at any time t. When does the mass first return to its equilibrium position?

(b) (13 points) If the mass is attached to a viscous damper with a damping constant 2 lb s/ft, and at $t = 0$ the mass is pushed up an additional 0.5 ft and released with a downward velocity of 0.5 ft/s, find the time τ such that $|u(t)| < 0.05$ ft for all $t > \tau$.

6. Consider a forced, undamped oscillator described by $u'' + \omega_0^2 u = 0.5\cos(\omega_f t)$, $u(0) = 0$, $u'(0) = 0$. Solutions for two different values of the forcing frequency ω_f are shown in figure 1, where t is in second, and u is in cm.

   (a) (7 points) First estimate the natural frequency ω_0 (you may use either top/bottom panel or both). Explain in detail how you calculate ω_0.

   (b) (8 points) Then estimate ω_f for both panels in terms of its relation with ω_0 (smaller, larger or equal). Explain in detail how you come up with your estimates. No credits could be earned without work to show/explain your answers.

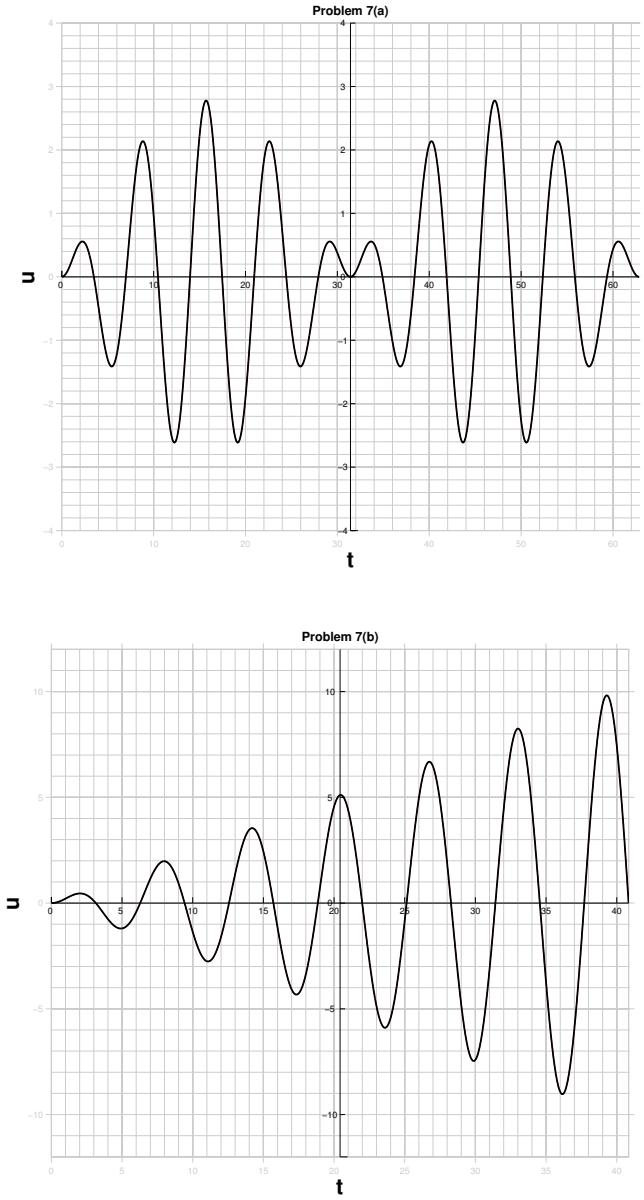

Figure 1: Solution u versus t for problem 7.