Math 222 Final Exam, May 6, 2016

Trigonometric identities: $\sin(A+B) = \sin(A)\cos(B)+\sin(B)\cos(A)$, $\cos(A+B) = \cos(A)\cos(B)-\sin(A)\sin(B)$. Closed-book, show all your work, all phones off and no calculator.

1. (10 points) Find the general solution for $\mathbf{x}(t)$ of the following 2×2 system:

$$\mathbf{x}(t)' = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \mathbf{x}. \tag{1}$$

Show that the solution $\mathbf{x}(t) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$ for any initial conditions.

2. (13 points) A function $f(x)$ is defined for $-\pi \leq x < \pi$ as

$$f(x) \quad = \quad \begin{cases} -\sin(x), & -\pi \leq x < 0, \\ \sin(x), & 0 \leq x < \pi, \end{cases} \tag{2}$$

with $f(x + 2\pi) = f(x)$.

(a) Sketch three periods of this function.

(b) Find the Fourier series of this function.

3. (10 points) Find the eigenvalues λ and eigenfunctions of the given boundary value problem (for a positive constant L)

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0. \tag{3}$$

4. (15 points) Solve the following initial value problems.

   (a) $ty' = \frac{1}{y+1}$, $y(1) = 0$.

   (b) $y' - y = e^{at}$, $y(0) = 0$ with a constant coefficient a. First find the solution when $a \neq 1$. Then find the solution when $a = 1$.

5. (17 points)

(a) Find the general solution of $y'' - 2y' - 3y = e^{-t}$.

(b) Given that $y_1(t) = e^t$ is a solution of $(t-1)y'' - ty' + y = 0$, find the general solution of this differential equation.

6. (10 points) Compute a power series solution around $x = 0$ for: $y'' + (x-1)y = 0$. Write out the first three non-zero terms in the series for the two fundamental solutions.

7. (15 points) Consider the initial value problem $y'' + 0.5y' + y = \delta(t-1)$, $y(0) = 0$, $y'(0) = 0$. Find the solution and plot its graph.

8. (10 points) Two tanks are connected with solution pouring from Tank 1 to Tank 2. At $t = 0$, Tank 1 originally contains 100 gal of fresh water and Tank 2 originally contains 100 gal of salty water with 10 lb of salt. Then water containing 0.5 lb of salt per gallon is poured into Tank 1 at 1 gal/min, and the mixture is allowed to leave at 1 gal/min, pouring into Tank 2. In Tank 2 the mixture is leaving also at 1 gal/min. First find the total amount of salt in Tank 1. Then find the total amount of salt in Tank 2 at any time.