Math 222 Exam 1, February 15, 2017

Read each problem carefully. Show all your work. Turn off your phones. No calculators!

1. (a) (5 points) Draw the direction field for the differential equation:

$$y' = (4 + y)(4 - y^2).$$

   (b) (5 points) Based on the direction field, describe the behavior of y as $t \to \infty$. For example, is there dependence of solution behavior on the initial value of y at $t = 0$? Sketch the solution with the initial condition $y(0) = 4$.

   (c) (5 points) Determine the order and type of the pendulum differential equation (g and l are constant coefficients): $\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$.

2. Consider the differential equation

$$y' = \frac{x(x^2 + 1)}{4y^3}.$$

   (5 points) First show that the differential equation is separable. (15 points) Then solve the differential equation with an initial condition $y(0) = -1/\sqrt{2}$, and express the solution in explicit form of x.

3. (a) (15 points) Find the solution of the given initial value problem, assuming a is a constant:

$$2y' - y = e^{t/3}, \quad y(0) = a.$$

   Find the value of a such that the solution has a critical point at $t = 6$.

   (b) (10 points) If the Wronskian of f and g is $2e^{-2t}$ and $f(t) = e^{-t}$, find $g(t)$ if $g(0) = 1$.

4. (15 points) A tank with a maximum volume of 120 liters originally contains 60 liters of water with 3 grams of salt dissolved in. Solution containing 0.25 grams of salt per liter enters the tank at the rate of 4 liters per hour. The well-mixed solution leaves the tank at the rate of 2 liters per hour. Let $Q(t)$ be the amount of salt in the tank (in grams) at time t (in hours), and $V(t)$ be the volume of solution (in liters).

   (a) Set up an initial value problem for the volume $V(t)$ and total salt amount $Q(t)$.

   (b) Find an expression for $V(t)$ and $Q(t)$ by solving the initial value problem in (a).

   (c) Find the total amount of salt when the tank is full.

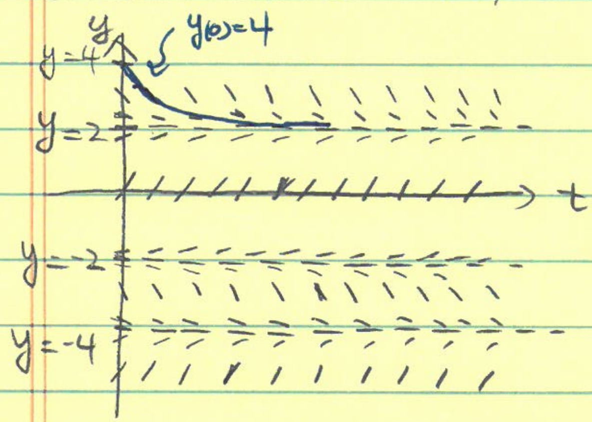
5. (10 points) Consider the initial value problem: $y' = \frac{t}{y}$, $y(0) = 1$. First use the Euler method to find an approximate value of the solution at $t = 0.2$ with a time step $h = 0.1$. (5 points) Find the exact solution, and then compute the difference between the exact solution and the Euler approximation at $t = 0.2$. Note that $\sqrt{1 + 0.2^2} \approx 1.0198$.

6. (10 points) Find a differential equation whose characteristic equation has roots $r_1 = -1$ and $r_2 = 0$. Write down the general solution for this differential equation, and let this general solution satisfy $y(0) = \alpha$ and $y'(0) = \beta$. Find conditions on α and β such that the solution approaches 1 as $t \to \infty$.

Grading Guidelines for Exam I, Feb 15, 2017. Math 222

Problem 1: $y' = (4+y)(4-y^2)$

(a) $y' = 0 = (4+y)(4-y^2)$, $4+y=0$, $4-y^2=0$, $y=-4, \pm 2$ $\boxed{3\text{pts}}$



$y' < 0$

$y' > 0$ $\boxed{2\text{pts}}$ for the sketch of direction field

$y' < 0$

$y' > 0$

(b) if $y_0 > 2$, $y \to 2$ as $t \to \infty$ $y(0)=4$, solution in blue curve. $\boxed{2\text{pts}}$

if $-2 < y_0 < 2$, $y \to 2$ as $t \to \infty$ $\boxed{3\text{pts}}$

if $y_0 < -2$ $y \to -2$ as $t \to \infty$

(c) $\theta'' + \frac{g}{\ell}\sin\theta = 0$ 2nd order $\boxed{2\text{pts}}$, non-linear $\boxed{3\text{pts}}$ differential equation

Problem 2 $y' = \frac{x(x^2+1)}{4y^3}$ $4y^3y' = x(x^2+1) \Rightarrow$ separable $\boxed{5\text{pts}}$

$$\int 4y^3\,dy = \int x(x^2+1)\,dx$$

$$y^4 = \frac{(x^2+1)^2}{4} + C$$

$y(0) = -\frac{1}{\sqrt{2}}$, $y^4(0) = \left(-\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4} = \frac{(0+1)^2}{4} + C$, $C=0$

$\boxed{10\text{pts}}$

$y^4 = \frac{(x^2+1)^2}{4}$, $y^2 = \pm\frac{x^2+1}{2}$, $y^2 = \frac{x^2+1}{2}$, $y = \pm\sqrt{\frac{x^2+1}{2}}$

$y = -\sqrt{\frac{x^2+1}{2}}$ because $y(0) = -\frac{1}{\sqrt{2}}$ $\boxed{5\text{pts}}$

Problem 3 (a) $2y' - y = e^{t/3}$, $y(0) = a$

$y' - \frac{1}{2}y = \frac{e^{t/3}}{2}$, $\mu = e^{\int -\frac{1}{2}dt} = e^{-t/2}$ $\boxed{5\text{pts}}$

$$y = \frac{\int \frac{e^{t/3}}{2} e^{-t/2}\,dt + c}{e^{-t/2}} = \frac{\frac{1}{2}\int e^{-\frac{t}{6}}\,dt + c}{e^{-t/2}} = \frac{\frac{1}{2}\left(\frac{1}{-\frac{1}{6}}e^{-\frac{t}{6}}\right) + c}{e^{-t/2}}$$

P.02

$$y = \frac{-3e^{-t/6} + c}{e^{-t/2}} = -3 \cdot e^{-\frac{t}{6}+\frac{t}{2}} + c \cdot e^{t/2} = -3e^{\frac{t}{3}} + c \cdot e^{t/2}$$

$$y(0) = a = -3 + c, \quad c = a+3$$

$$y = -3e^{t/3} + (a+3)e^{t/2} \quad \boxed{5pts}$$

$$y' = -e^{t/3} + \frac{a+3}{2}e^{t/2}$$

$$y'(6) = 0 = -e^{2} + \frac{a+3}{2}e^{3} \qquad \frac{a+3}{2} = e^{-1} \qquad \boxed{a = 2e-3} \quad \boxed{5pts}$$

(b) $fg' - f'g = 2e^{-2t}, \quad f = e^{-t}, \quad g(0) = 1 \quad \boxed{5pts}$

$$e^{-t}g' + e^{-t}g = 2e^{-2t}, \quad g' + g = 2e^{-t}, \quad g(0) = 1$$

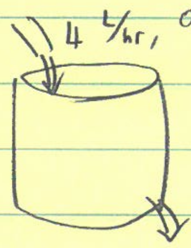
$$\mu = e^{\int 1\,dt} = e^{t}, \qquad g = \frac{\int 2e^{-t}e^{t}\,dt + c}{e^{t}} = \frac{2t + c}{e^{t}} = 2te^{-t} + ce^{-t}$$

$$g(0) = 1 = c, \quad g(t) = 2te^{-t} + e^{-t} \quad \boxed{5pts}$$

Problem 4

$V(0) = 60L$

$Q(0) = 3g$



$$V' = 4 - 2 = 2, \qquad V = 2t + V_0 = 2t + 60$$

$\boxed{3pts}$

$$Q' = 4 \times 0.25 - \frac{Q}{2t+60} \times 2$$

$$Q' + \frac{Q}{t+30} = 1 \quad \boxed{3pts}$$

$$\mu = e^{\int \frac{1}{t+30}dt} = t+30$$

$$Q = \frac{\int (t+30)\cdot 1 \cdot dt + c}{t+30} = \frac{1}{2}\frac{(t+30)^2}{t+30} + \frac{c}{t+30}$$

$$Q(0) = 3 = \frac{1}{2} \cdot \frac{30^2}{30} + \frac{c}{30} = 15 + \frac{c}{30}, \qquad -12 = \frac{c}{30}, \quad c = -360$$

$$Q = \frac{1}{2}(t+30) - \frac{360}{t+30} \ (g) \quad \boxed{4pts}$$

$$V = 120 \quad \text{when} \quad t = \frac{120-60}{2} = 30 \text{ hr} \quad \boxed{2pts}$$

$Q(30) = \frac{1}{2}(30+30) - \frac{360}{60} = 30 - 6 = 24\ (g)$ $\boxed{3pts}$

Problem 5:

$y' = \frac{t}{y}$

$y_{n+1} = y_n + h \cdot \frac{t_n}{y_n}$, $y_1 = 1$, $t_1 = 0$ $\boxed{5pts}$

$y_2 = y_1 + 0.1 \times \frac{t_1}{y_1} = 1 + 0.1 \times \frac{0}{1} = 1$ at $t = 0.1$ $\boxed{2pts}$

$y_3 = y_2 + 0.1 \times \frac{t_2}{y_2} = 1 + 0.1 \times \frac{0.1}{1} = 1.01$ at $t = 0.2$ $\boxed{3pts}$

$y y' = t$

$y\,dy = t\,dt$, $\frac{y^2}{2} = \frac{t^2}{2} + c$, $y^2 = t^2 + C$, $y(0) = 1$, $C = 0$

$y = \sqrt{t^2+1}$

$y(0.2) = \sqrt{1+0.2^2} \approx 1.0198$

difference $\Rightarrow$ $1.0198 - 1.01 = 0.0098$ $\boxed{5pts}$

Problem 6

$(r+1)r = 0$ $r^2 + r = 0$, $y'' + y' = 0$ $\boxed{5pts}$

$y = c_1 + c_2 e^{-t}$, $y' = -c_2 e^{-t}$

$\boxed{2pts}$

$y(0) = \alpha = c_1 + c_2$ $c_1 = \alpha - c_2 = \alpha - (-\beta) = \alpha + \beta$

$y'(0) = \beta = -c_2$

$y = \alpha + \beta - \beta e^{-t}$

$y \to 1$ as $t \to \infty$, $\boxed{\alpha + \beta = 1}$ $\boxed{3pts}$