Math 222 Exam 2, March 8, 2017

Read each problem carefully. Show all your work. Turn off your phones. No calculators!

1. (15 points) Solve the IVP:

$$y'' + 2y' + 5y = 0,\ y(0) = 5,\ y'(0) = 7.$$

   Describe the solution behavior as $t \to \infty$.

2. Consider the IVP

$$t^2y'' + 5ty' + 4y = 0,\ y(1) = 0,\ y'(1) = 1.$$

   (a) (5 points) Show that $y_1 = t^{-2}$ is a solution to the above equation.

   (b) (10 points) Find a second independent fundamental solution, and then solve the IVP.

3. (a) (8 points) Find the general solution to the differential equation

$$y'' - 2y' - 3y = 3te^{2t}.$$

   (b) (7 points) Find a suitable form for the solution of the differential equation using the method of undetermined coefficients but DO NOT solve for the coefficients:

$$y'' - 5y' + 6y = e^t \cos 2t + e^{2t}(3t + 4)\sin t + 12t^2e^{-t}.$$

4. (15 points) Solve the IVP

$$y'' + 2y' + y = t^{-1}e^{-t},\ y(1) = 1,\ y'(1) = 0.$$

   Describe the solution behavior as $t \to \infty$.

5. A mass weighing 4 lb is attached to a vertical spring of constant $k = 2$ lb/ft. After the mass comes to rest, it is pulled down 1 foot and released with an upward velocity of 4 ft/s. In these units the gravitational acceleration is $g = 32$.

   (15 points) If there is no damping, determine the position $u(t)$ of the mass at any time. Find the amplitude, period and phase angle of the motion. Note that

$$\cos(B - C) = \cos B \cos C + \sin B \sin C.$$

6. Consider a forced vibration system described by the IVP

$$u'' + u = \cos(\omega t), \; u(0) = 0, \; u'(0) = 0.$$

(a) (8 points) Find the solution $u(t)$ when $\omega = 1$.

(b) (8 points) Find the solution $u(t)$ when $\omega = 0.9$.

7. (9 points) Figures 1, 2 and 3 are sketches of solutions of second order differential equations. Based on the solution behavior, identify the differential equation(s) in the list that may correspond to the given solution(s) in each figure. You DO NOT have to solve the differential equations, but you do have to explain your answers using mathematical reasoning.

$$\begin{array}{ll} (a)\; y'' + 4y' + 4y = 0, & (b)\; y'' + 4y' + 4y = t, \\ (c)\; y'' + y' + 9.25y = 0, & (d)\; y'' + 9y = \cos 3t, \\ (e)\; y'' + y' + 9.25y = \sin 3t, & (f)\; y'' + 9y = 0, \\ (g)\; y'' - y' - 2y = 0, & (h)\; y'' - y' - 2y = e^{2t}. \end{array}$$

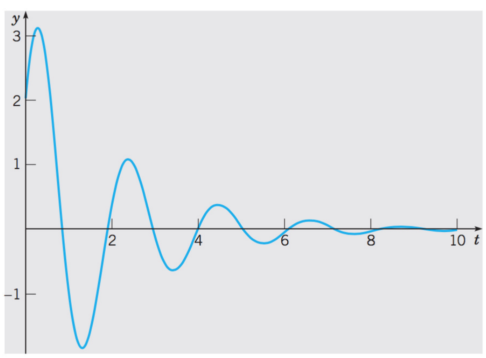

Figure 1

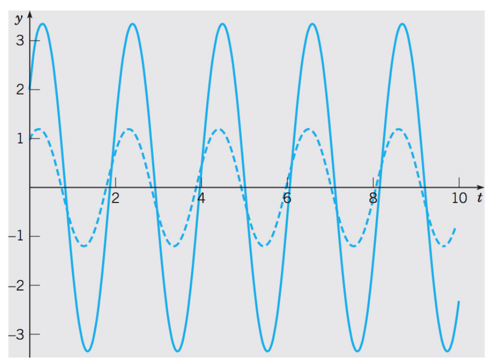

Figure 2

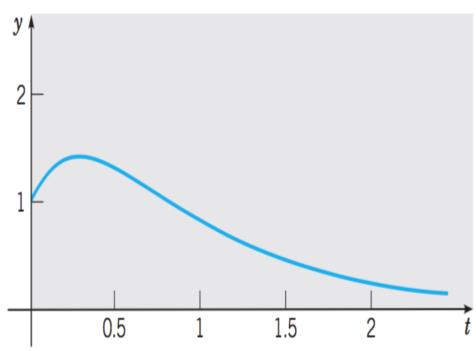

Figure 3

Grading Guidelines for Exam II, March 08, 2017, Math 222

Problem 1: $y''+2y'+5y=0, \quad y(0)=5, \; y'(0)=7$

$r^2+2r+5=0$

$(r+1)^2+2^2=0, \quad r=-1\pm 2i$

$y=e^{-t}(c_1\cos 2t + c_2 \sin 2t)$ 4 pts

$y' = -e^{-t}(c_1\cos 2t + c_2\sin 2t) + e^{-t}(-2c_1\sin 2t + 2c_2\cos 2t)$

$y(0) = c_1 = 5$ 4pts

$y'(0) = -5 + 2c_2 = 7, \quad 2c_2=12, \quad c_2=6$ 4pts

$$\boxed{y(t)=e^{-t}(5\cos 2t + 6\sin 2t)}$$

$y\to 0$ as $t\to\infty$ 3pts

Problem 2: $t^2y''+5ty'+4y=0, \quad y(1)=0, \; y'(1)=1$

(a) $y_1=t^{-2}, \; y_1'=-2t^{-3}, \; y_1''=6t^{-4}$

$t^2\cdot 6t^{-4}+5t\cdot(-2t^{-3})+4t^{-2}$

$=6t^{-2}-10t^{-2}+4t^{-2}=0$ ✓ y_1 is a solution 5pts

(b) $y_2=vy_1, \quad y_2'=v'y_1+vy_1', \quad y_2''=v''y_1+2v'y_1'+vy_1''$

$t^2(v''y_1+2v'y_1'+vy_1'')+5t(v'y_1+vy_1')+4vy_1=0$

$t^2y_1v''+(2t^2y_1'+5ty_1)v'=0$

$y_1=t^{-2}$,

$v''+(2t^2(-2t^{-3})+5t\cdot t^{-2})v'=0$

$v''+\frac{1}{t}v'=0$

$\frac{v''}{v'}=-\frac{1}{t}, \quad \ln v' = -\ln t = \frac{1}{t}, \quad v'=\frac{1}{t} \quad v=\ln t$ 3pts

$y_2=t^{-2}\ln t$, 3pts $\quad y=c_1t^{-2}+c_2t^{-2}\ln t, \quad y(1)=0=c_1$ 2pts

$y'=-2c_1t^{-3}-2c_2t^{-3}\ln t+c_2t^{-3} \quad y'(1)=c_2=1$ 2pts

$$\boxed{y=t^{-2}\ln t}$$

Problem 3 (a) $y''-2y'-3y = 3te^{2t}$

$r^2-2r-3=0$, $(r-3)(r+1)=0$, $r=-1,3$

$y_c = C_1e^{-t}+C_2e^{3t}$ [2pts]

[3pts]

$Y=(At+B)e^{2t}$, $Y' = Ae^{2t}+2(At+B)e^{2t} = (2At+2B+A)e^{2t}$

$Y'' = 2Ae^{2t} + 2(2At+A+2B)e^{2t}$

$Y''-2Y'-3Y = (4At+4A+4B)e^{2t} - 2(2At+2B+A)e^{2t} - 3(At+B)e^{2t}$

$= 3te^{3t}$

$4At-4At-3At=3t$ $\quad A=-1$

$4A+4B-4B-2A-3B = 2A-3B=0$, $\quad B=\frac{2A}{3}=-\frac{2}{3}$

$\boxed{y = C_1e^{-t}+C_2e^{3t}+(-t-\frac{2}{3})e^{2t}}$ [3pts]

(b) $y''-5y'+6y = e^{2t}\cos 2t + e^{2t}(3t+4)\sin t + 12t^2e^{-t}$

$r^2-5r+6=0$, $(r-3)(r-2)=0$, $r=2,3$

$y_c = C_1e^{2t}+C_2e^{3t}$ [2pts]

$Y = (A\cos 2t + B\sin 2t)e^{2t} + (Ct+D)\sin t\, e^{2t} + (Gt^2+Ht+I)e^{-t}$
$+ (Et+F)\cos t\, e^{2t}$ [5pts]

Problem 4 $y''+2y'+y = t^{-1}e^{-t}$, $y(1)=1$, $y'(1)=0$

$r^2+2r+1=0$, $r=-1,-1$ $\quad y_c = C_1e^{-t}+C_2te^{-t}$ [5pts]

$W[y_1,y_2] = \begin{vmatrix} e^{-t} & te^{-t}\\ -e^{-t} & e^{-t}-te^{-t}\end{vmatrix} = e^{-2t}$

$Y = \left(\int \frac{-te^{-t}\cdot t^{-1}e^{-t}}{e^{-2t}}dt\right)e^{-t} + \left(\int\frac{e^{-t}t^{-1}e^{-t}}{e^{-2t}}dt\right)te^{-t}$

$= -te^{-t} + \ln t\cdot te^{-t}$

$y = C_1e^{-t}+C_2te^{-t}-te^{-t}+\ln t\cdot t\cdot e^{-t} = C_1e^{-t}+C_2te^{-t}+t\ln t\, e^{-t}$ [5pts]

$y' = -C_1e^{-t}+C_2e^{-t}-C_2te^{-t}+\ln t\, e^{-t}+e^{-t}-t\ln t\, e^{-t}$

$y(1) = C_1e^{-1}+C_2e^{-1}=1$, $\quad y'(1) = (-C_1+C_2-C_2)e^{-1}+e^{-1}=0$

$c_1 + c_2 = e$

$-c_1 + 1 = 0$

$c_1 = 1, \quad c_2 = e - c_1 = e - 1$

$$\boxed{y(t) = e^{-t} + (e-1)te^{-t} + t\ln t\, e^{-t}}$$ 5pts

Problem 5

$m = 4/32 = \frac{1}{8}$

$mu'' + ku = 0, \quad \frac{1}{8}u'' + 2u = 0, \quad u'' + 16u = 0, \quad u(0) = 1,\ u'(0) = -4$ 5pts

$u = c_1\cos 4t + c_2 \sin 4t \qquad u(0) = 1 = c_1$

$u' = -4c_1 \sin 4t + 4c_2 \cos 4t \qquad u'(0) = -4 = 4c_2, \quad c_2 = -1$

$u(t) = \cos 4t - \sin 4t$ 5pts

$u(t) = \sqrt{2}\cdot\cos(4t - \delta), \qquad \delta = \tan^{-1}(-1) = \frac{-\pi}{4}$

amplitude $= \sqrt{2}$

frequency $= 4$ 5pts

phase $= \frac{-\pi}{4}$

Problem 6 (a) $u'' + u = \cos t$

$u_c = c_1 \cos t + c_2 \sin t, \qquad U = (A\cos t + B\sin t)t$ 4pts

$U' = (A\cos t + B\sin t) + (-A\sin t + B\cos t)t$

$U'' = -A\sin t + B\cos t - A\sin t + B\cos t$
$+ (-A\cos t - B\sin t)t$

$U'' + U = -(A\cos t + B\sin t)t - 2A\sin t + 2B\cos t + (A\cos t + B\sin t)t = \cos t$

$2B = 1, \quad B = \frac{1}{2} \qquad u(t) = c_1\cos t + c_2 \sin t + \frac{t}{2}\sin t$

$2A = 0, \quad A = 0$

$u'(t) = -c_1\sin t + c_2\cos t + \frac{1}{2}\sin t + \frac{t}{2}\cos t$

$u(0) = 0 = c_1$

$u'(0) = 0 = c_2 \qquad \boxed{u(t) = \frac{t}{2}\sin t}$ 4pts

(b) $u'' + u = \cos 0.9t$

$U = A\cos 0.9t + B\sin 0.9t$ 4pts

$(-0.9^2 + 1)A\cos 0.9t + (-0.9^2 + 1)B\sin 0.9t = \cos 0.9t$

$B = 0, \quad A = \frac{1}{1 - 0.81} = \frac{1}{0.19}, \quad u = c_1\cos t + c_2\sin t + \frac{1}{0.19}\cos 0.9t$

$u' = -C_1 \sin t + C_2 \cos t - \frac{0.9}{0.19} \sin 0.9t$

$u(0) = 0 = C_1 + \frac{1}{0.19}$, $C_1 = -\frac{1}{0.19}$

$u'(0) = 0 = C_2 = 0$

$\boxed{u = -\frac{1}{0.19}\cos t + \frac{1}{0.19}\cos 0.9t}$ 4pts

Problem 7: ① figure 1 ⇒ $y \sim e^{-\alpha t}\cdot(C_1\cos\beta t + C_2 \sin\beta t)$ w/ $\alpha > 0$ ⇒ c or e

solution also decays to zero ⇒ cannot be e

$\boxed{\text{Figure 1} \to C}$ 3pts

② Figure 2 ⇒ oscillatory solution could be (d) → resonnance, (f), (e) w/ forcing

⇒ $\boxed{\text{Figure 2 can correspond to e, f}}$ 3pts

③ Figure 3 ⇒ transient growth + decay to zero.

Either a or b → b has a linear polynomial due to forcing

⇒ $\boxed{\text{Figure 3 can correspond to a}}$. 3pts