Solutions for HWK week 13

Problem 1 (problem 21 on page 418)

$$tx' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} x$$

$$x = t^{\xi} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad tx' = \xi t^{\xi} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t^{\xi} \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} -1-\xi & -1 \\ 2 & -1-\xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(-1-\xi)^2 + 2 = 0, \quad (\xi+1)^2 = -2, \quad \xi = -1 \pm \sqrt{2}i$$

$$\xi = -1+\sqrt{2}i, \quad \begin{pmatrix} -\sqrt{2}i & -1 \\ 2 & \sqrt{2}i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -\sqrt{2}i\, x_1 - x_2 = 0$$

$$\frac{x_1}{x_2} = -\frac{1}{\sqrt{2}i} = \frac{\sqrt{2}i}{1}$$

Corresponding eigenvector $= \begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix}$

$$\xi = -1-\sqrt{2}i \qquad \begin{pmatrix} \sqrt{2}i & -1 \\ 2 & \sqrt{2}i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \sqrt{2}i\, x_1 - x_2 = 0$$

$$\frac{x_1}{x_2} = \frac{1}{\sqrt{2}i} = -\frac{\sqrt{2}i}{1}$$

corresponding eigenvector $= \begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix}$

solution $\Rightarrow$ $x = c_1 t^{-1+\sqrt{2}i} \begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix} + c_2 t^{-1-\sqrt{2}i} \begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix}$

$$t^{-1+\sqrt{2}i} = t^{-1} \cdot t^{\sqrt{2}i} = t^{-1} \cdot \left(\cos(\sqrt{2}\ln t) + i \sin(\sqrt{2}\ln t) \right)$$

general solution can also be expressed as

$$x = t^{-1} \cdot \left[\bar{c}_1 \cdot \left(\cos(\sqrt{2}\ln t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin(\sqrt{2}\ln t) \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \right) + \bar{c}_2 \left(\cos(\sqrt{2}\ln t) \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \sin(\sqrt{2}\ln t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right]$$

Problem 2 (problem 22 page 596)

$$EI \cdot \frac{d^4y}{dx^4} = f(x), \quad \frac{d^4y}{dx^4} = \frac{f(x)}{EI} = k = -1$$

(a) $y(0) = y''(0) = y(L) = y''(L) = 0$

$$y = a + bx + cx^2 + dx^3 - \frac{1}{4!}x^4$$

$$y'' = 2c + 6dx - \frac{4\times 3}{4!}x^2 = 2c + 6dx - \frac{1}{2}x^2$$

P.02

$$y(0) = a = 0$$
$$y(L) = 0 = a + bL + cL^2 + dL^3 - \frac{1}{4!}L^4$$
$$y''(0) = 2c = 0$$
$$y''(L) = 2c + 6dL - \frac{1}{2}L^2 = 0$$

$\Rightarrow$ $bL + dL^3 = \frac{1}{24}L^4$

$6dL = \frac{1}{2}L^2$

$d = \frac{1}{12}L$

$bL + \frac{1}{12}L^2 = \frac{1}{24}L^4$

$b = \frac{1}{24}L^3 - \frac{1}{12}L$

$= \frac{L}{24}(L^2-2)$

$$\boxed{y = \frac{L}{24}(L^2-2)\,x + \frac{1}{12}L\cdot x^3 - \frac{1}{24}x^4}$$

(b) $y(0) = y'(0) = y(L) = y'(L) = 0$.

$y(0) = a = 0$ $\qquad$ $y(L) = a + bL + cL^2 + dL^3 - \frac{1}{4!}L^4 = 0$

$y'(0) = b = 0$ $\qquad$ $y'(L) = b + 2cL + 3dL^2 - \frac{1}{6}L^3 = 0$

$cL^2 + dL^3 = \frac{1}{24}L^4$, $\qquad$ $c + dL = \frac{1}{24}L^2$ $\qquad$ $dL = \frac{1}{12}L^2$

$2c + 3dL = \frac{1}{6}L^2$ $\qquad$ $\boxed{d = \frac{1}{12}L}$

$c = \frac{1}{24}L^2 - \frac{1}{12}L^2 = \frac{-L^2}{24}$

$$\boxed{y(x) = -\frac{L^2}{24}x^2 + \frac{L}{12}x^3 - \frac{1}{24}x^4}$$

(c) $y(0) = 0$, $y'(0) = 0$, $y''(L) = 0$, $y'''(L) = 0$

$y(0) = a = 0$

$y'(0) = b = 0$

$y''(L) = 2c + 6dL - \frac{3}{6}L^2 = 0$ $\qquad$ $L^2 - \frac{1}{2}L^2 + 2c = 0$

$y'''(L) = 6d - L = 0$ $\qquad$ $d = \frac{L}{6}$ $\qquad$ $c = \frac{L^2}{4}$

$$\boxed{y(x) = \frac{L^2}{4}x^2 + \frac{L}{6}x^3 - \frac{1}{24}x^4}$$

Problem 3 (problem 29 page 606)

(a) $\underline{u} = a_1\underline{V_1} + a_2\underline{V_2} + a_3\underline{V_3}$, $\qquad$ $\underline{V_1}\cdot\underline{V_2} = 0 = \underline{V_1}\cdot\underline{V_3} = \underline{V_2}\cdot\underline{V_3}$

$\therefore\ a_1 \cdot \underline{V_1}\cdot\underline{V_1} = \underline{u}\cdot\underline{V_1}$, $\quad \boxed{a_1 = \frac{\underline{u}\cdot\underline{V_1}}{\underline{V_1}\cdot\underline{V_1}}}$ $\quad a_i = \frac{\underline{u}\cdot\underline{V_i}}{\underline{V_i}\cdot\underline{V_i}}$ $\quad i = 1, 2, 3$

(b) From equation 10, $\int_{-L}^{L} f \cos\frac{n\pi x}{L}\,dx = (f, \varphi_n)$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx = (\phi_0, \phi_n), \quad \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = (\phi_m, \phi_n)$$

eq. 10 $\Rightarrow$ $(f, \phi_n) = \frac{a_0}{2}(\phi_0, \phi_n) + \sum_{m=1}^{\infty} a_m (\phi_m, \phi_n) + \sum_{m=1}^{\infty} b_m (\psi_m, \phi_n)$

(c) because of orthogonality between ϕ_n & ψ_n,

we obtain

$$a_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}, \quad b_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}$$

Problem 4 $\quad y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$

$\lambda = 0$ $\quad y'' = 0, \quad y = ax + b \quad y'(0) = a = 0, \quad y(L) = b = 0$

$y = 0 \Rightarrow \lambda = 0$ is not an eigenvalue

$\lambda > 0$ $\quad y'' + \lambda y = 0, \quad r = \pm\sqrt{\lambda}\, i, \quad y_1 = \cos\sqrt{\lambda}x, \; y_2 = \sin\sqrt{\lambda}x,$

$y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x, \quad y' = \sqrt{\lambda}\, c_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}\, c_2 \cos\sqrt{\lambda}x$

$y'(0) = \sqrt{\lambda} \cdot c_2 \cdot 1 = 0 \qquad c_2 = 0$

$y(L) = c_1 \cos\sqrt{\lambda}L = 0 \qquad \sqrt{\lambda}L = \frac{\pi}{2} + n\pi, \quad \sqrt{\lambda} = \frac{\frac{\pi}{2} + n\pi}{L}$

$$\lambda_n = \left(\frac{2n+1}{2L}\pi\right)^2$$

$\lambda < 0$ $\quad y'' - |\lambda| y = 0, \quad y_1 = e^{\sqrt{|\lambda|}x}, \quad y_2 = e^{-\sqrt{|\lambda|}x}$

$y = c_1 e^{\sqrt{|\lambda|}x} + c_2 e^{-\sqrt{|\lambda|}x}$

$y' = \sqrt{|\lambda|} \cdot \left(c_1 e^{\sqrt{|\lambda|}x} - c_2 e^{-\sqrt{|\lambda|}x}\right)$

$y'(0) = \sqrt{|\lambda|}\,(c_1 - c_2) = 0$

$y(L) = c_1 e^{\sqrt{|\lambda|}L} + c_2 e^{-\sqrt{|\lambda|}L} = 0 \quad \Rightarrow c_1 = c_2 = 0 \quad \lambda < 0$ is not an eigenvalue