P.01

Solution to Homework #2

Problem 1: $ty' + (t+1)y = 2te^{-t}, \quad y(1) = a$

$$y' + \frac{t+1}{t}y = 2e^{-t},$$

$$\mu = e^{\int \frac{t+1}{t}dt} = e^{t+\ln t} = te^{t}$$

$$y = \frac{\int 2e^{-t}\cdot te^{t}dt + c}{te^{t}} = \frac{\int 2t\,dt + c}{te^{t}} = \frac{t^2+c}{te^{t}}$$

$$y = te^{-t} + c\frac{1}{t}\cdot e^{-t}$$

as $t \to \infty$, $y \to \infty$ regardless of c.

y touches the time axis $(y=0)$ at $t=2$

$$\Rightarrow y(2) = 0 = 2e^{-2} + \frac{c}{2}e^{-2}, \quad \boxed{c = -4}$$

Solution reaches a critical point at $t = 1/2$

$$\Rightarrow y' = 0 \ @ \ t = 1/2$$

$$y' = -\frac{t+1}{t}y + 2e^{-t}$$

$$y'(\tfrac{1}{2}) = -\frac{3/2}{1/2}y(1/2) + 2e^{-1/2} = 0$$

$$-3y(1/2) = -2e^{-1/2}, \quad y(\tfrac{1}{2}) = \frac{2}{3}e^{-1/2}$$

$$y(1/2) = \frac{1}{2}e^{-1/2} + 2ce^{-1/2} = \frac{2}{3}e^{-1/2}$$

$$\frac{1}{2} + 2c = \frac{2}{3}, \quad 2c = \frac{4-2}{6} = \frac{2}{6} = 1/3, \quad \boxed{c = \frac{1}{6}}$$

Problem 2 ① Problem 27 on page 49

$y' = ty(4-y)/3, \quad y(0) = y_0, \quad t > 0$

at equilibrium $y' = 0 \quad \frac{t}{3}y(4-y) = 0, \quad y = 0, 4$

$y' > 0$ when $0 < y < 4$

$y' < 0$ when $4 < y$

$y' < 0$ when $y < 0$

ⓐ $y \to 4$ as $t \to \infty$ when $y_0 > 0$

$y \to -\infty$ as $t \to \infty$ when $y_0 < 0$

ⓑ $\frac{dy}{dt} = \frac{t}{3} y(4-y),$

$\frac{dy}{y(4-y)} = \frac{t}{3}dt, \quad \int \frac{dy}{y(4-y)} = \int \frac{t}{3}dt = \frac{t^2}{6} + C$

$\int \frac{dy}{y(4-y)} = \int \frac{1}{4}\left(\frac{1}{y} + \frac{1}{4-y}\right) dy = \frac{1}{4} \int \frac{1}{y} + \frac{1}{4-y} dy$

$= \frac{1}{4} \ln\left(\frac{|y|}{|4-y|}\right) = \frac{t^2}{6} + C$

$\ln\left|\frac{y}{4-y}\right| = \frac{2t^2}{3} + C, \quad \left|\frac{y}{4-y}\right| = e^{\frac{2t^2}{3}+C} = A \cdot e^{\frac{2t^2}{3}},$

if $y_0 = 1/2$, y remains positive for all time, and $y < 4$ for all time

therefore $\left|\frac{y}{4-y}\right| = \frac{y}{4-y} = Ae^{\frac{2t^2}{3}}$

$y = 4Ae^{\frac{2t^2}{3}} - Ae^{\frac{2t^2}{3}}y, \quad \left(1 + Ae^{\frac{2t^2}{3}}\right)y = 4Ae^{\frac{2t^2}{3}},$

$y(t) = \frac{4Ae^{2t^2/3}}{1+Ae^{2t^2/3}}$; $y(0) = \frac{1}{2}$, $\frac{1}{2} = \frac{4A}{1+A}$, $8A = 1+A$ $A = \frac{1}{7}$

$y(T) = 3.98 = \frac{4e^{2T^2/3}}{7+e^{2T^2/3}}$, $7 \times 3.98 = 0.02\, e^{2T^2/3}$

$\frac{7\times 398}{2} = e^{2T^2/3}$, $\ln(7\times 199) = 2T^2/3$,

$T^2 = \frac{3}{2}\ln(7\times 199)$, $\boxed{T = \sqrt{\frac{3}{2}\ln(7\times 199)}}$

② Problem 28 on page 49

$y' = \frac{t}{1+t}\, y(4-y)$, $y(0) = y_0 > 0$

as t increases from zero (initial value)

$y' = \frac{t}{1+t}\, y(4-y) \; \begin{matrix} > \\ = \\ < \end{matrix} \; 0$ when $\begin{matrix} y(4-y) > 0 \\ y(4-y) = 0 \\ y(4-y) < 0 \end{matrix}$

$y > 4$ $y' < 0$

$0 < y < 4$ $y' > 0$

$y < 0$ $y' < 0$

ⓐ if $y_0 > 0$, $y \to 4$ as $t \to \infty$ from above

ⓑ $\frac{dy}{dt} = \frac{t}{1+t}\, y(4-y)$, $\frac{dy}{y(4-y)} = \frac{t}{1+t}\,dt$, $\frac{1}{4}\ln\left|\frac{y}{4-y}\right| = \int \frac{t}{1+t}\,dt$

$\frac{1}{4}\ln\left|\frac{y}{4-y}\right| = \int 1 - \frac{1}{1+t}\,dt = t - \ln(1+t) + C$

$\ln\left|\frac{y}{4-y}\right| = 4t - 4\ln(1+t) + c$, $\left|\frac{y}{4-y}\right| = \frac{Ae^{4t}}{(1+t)^4}$, if $y_0 = 2$, $\frac{y}{4-y} > 0$ for all time,

therefore, $\frac{y}{4-y} = \frac{Ae^{4t}}{(1+t)^4}$, $y = (4-y)\frac{Ae^{4t}}{(1+t)^4}$,

$y_0 = 2$, $2 = 2 \cdot \frac{A \cdot e^0}{(1+0)^4}$, $A = 1$,

$y = (4-y)\frac{e^{4t}}{(1+t)^4}$, $\left(1 + \frac{e^{4t}}{(1+t)^4}\right) y = \frac{4e^{4t}}{(1+t)^4}$

$$y = \frac{4e^{4t}/(1+t)^4}{1 + \frac{e^{4t}}{(1+t)^4}} = \frac{4e^{4t}}{(1+t)^4 + e^{4t}}$$

$y(T) = 3.99 = \frac{4e^{4T}}{(1+T)^4 + e^{4T}}$, $3.99 \times (1+T)^4 = 0.01 e^{4T}$

$399 \times (1+T)^4 = e^{4T}$

$\boxed{T = 17.7039 \text{ from Matlab}}$

(c) $y = (4-y)\frac{Ae^{4t}}{(1+t)^4}$, $\left(1 + \frac{Ae^{4t}}{(1+t)^4}\right) y = \frac{4Ae^{4t}}{(1+t)^4}$,

$y = \frac{4Ae^{4t}}{(1+t)^4 + Ae^{4t}}$, need to find A such that

$3.99 < \frac{4Ae^{4T}}{(1+T)^4 + Ae^{4T}} < 4.01$ before $T = 2$

Note that from the direction field, $y' > 0$ for $0 < y < 4$
$y' < 0$ for $y > 4$

therefore y varies monotonically.

(y increases when $y < 4$
y decrease when $y > 4$)

P.05

$T=2$ $\quad 3.99 < \dfrac{4A\cdot e^8}{(1+2)^4+Ae^8}$, $\quad 3.99\times 3^4 < 0.01e^8 A$,

$$A > \frac{399\times 3^4}{e^8}$$

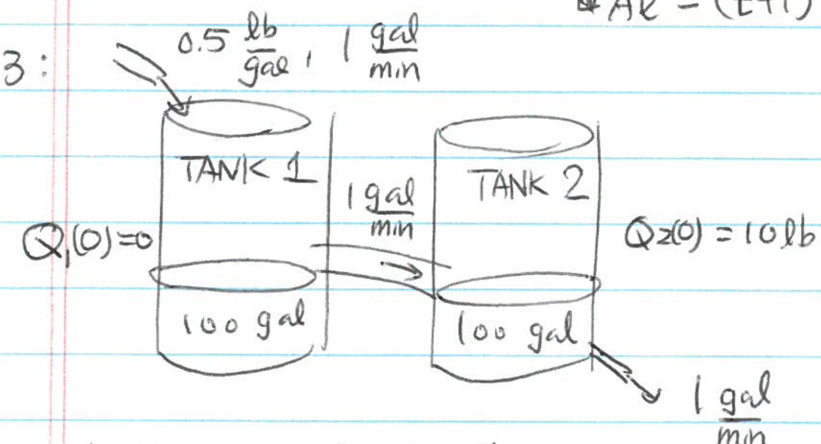
$\dfrac{4Ae^8}{-(1+2)^4+Ae^8} \overset{>}{} 4.01$, $\quad 4.01\times 3^4 \overset{>}{} 0.01\,Ae^8$

$$\frac{4.01}{0.01}\times\frac{3^4}{e^8} > A$$

$$\boxed{\frac{399\times 3^4}{e^8} < A < \frac{401\times 3^4}{e^8}}$$

the minus sign is here because when $y_0 > 4$, the solution is $y(t) = \dfrac{4Ae^{4t}}{Ae^{4t}-(t+1)^4}$

Problem 3:



$V_1(t) = 100$ gal at all time because $\dfrac{dV_1}{dt} = 1-1=0$, $V_1(0)=100$ gal

$V_2(t) = 100$ gal at all time $\quad \dfrac{dV_2}{dt} = 1-1=0 \quad V_2(0)=100$ gal

$$\frac{dQ_1}{dt} = \frac{1}{2}\cdot 1 - \frac{Q_1}{100}\cdot 1, \quad Q_1(0)=0$$

$$\frac{dQ_2}{dt} = \frac{Q_1}{100}\times 1 - \frac{Q_2}{100}\cdot 1, \quad Q_2(0)=10$$

$Q_1' + \dfrac{Q_1}{100} = \dfrac{1}{2}$, $\quad \mu = e^{\int\frac{1}{100}dt} = e^{\frac{t}{100}}$, $\quad Q_1 = \dfrac{\int \frac{1}{2}e^{\frac{t}{100}}dt + c}{e^{\frac{t}{100}}}$

$$Q_1 = \frac{50e^{\frac{t}{100}}+c}{e^{t/100}} = 50+ce^{-\frac{t}{100}}$$

$Q_1(0) = 50+c = 0$, $\quad c = -50$

P.06

$$\boxed{Q_1(t) = +50 - 50e^{-t/100} = 50(1-e^{-t/100}) \text{ lb}}$$

$$Q_2'(t) = \frac{50(1-e^{-t/100})}{100} - \frac{Q_2}{100}, \quad Q_2(0) = 10$$

$$Q_2' + \frac{Q_2}{100} = \frac{1}{2}(1-e^{-t/100}), \qquad Q_2 = \frac{\int \frac{1}{2}(1-e^{-t/100})e^{\frac{t}{100}}\,dt + c}{e^{\frac{t}{100}}}$$

↳ same integrating factor as for Q_1

$$= \frac{\int \frac{1}{2}e^{\frac{t}{100}} - \frac{1}{2}\,dt + c}{e^{t/100}}$$

$$= \frac{50e^{\frac{t}{100}} - \frac{1}{2}t + c}{e^{\frac{t}{100}}}$$

$$Q_2(t) = 50 - \frac{1}{2}te^{-\frac{t}{100}} + ce^{-\frac{t}{100}}$$

$$Q_2(0) = 10 = 50 + c, \quad c = -40$$

$$\boxed{Q_2(t) = 50 - 40e^{-\frac{t}{100}} - \frac{1}{2}te^{-\frac{t}{100}}} \text{ lb}$$