Solution to Extra Homework for Week 3

Problem 1

$y' = 2y - 1, \quad y(0) = 1$

$y' - 2y = -1, \quad \mu = e^{\int -2\,dt} = e^{-2t}, \quad y = \dfrac{\int -1 \cdot e^{-2t}\,dt + c}{e^{-2t}} = \dfrac{\frac{1}{2}e^{-2t} + C}{e^{-2t}}$

$y = \frac{1}{2} + c \cdot e^{2t}, \quad y(0) = 1 = \frac{1}{2} + c \cdot e^{0} \Rightarrow c = \frac{1}{2}$

$y = \frac{1}{2}(1 + e^{2t})$

$y_{n+1} - y_n = h \cdot (2y_n - 1)$

$y_{n+1} = y_n + h(2y_n - 1)$

$y_0 = 1, \quad t_0 = 0$

$h = 0.1$
$y_1 = y_0 + 0.1 \times (2 \times 1 - 1) = 1 + 0.1 \times 1 = 1.1$
$y_2 = y_1 + 0.1 \times (2 \times 1.1 - 1) = 1.1 + 0.1 \times (1.2) = 1.22$

$h = 0.05$
$y_1 = y_0 + 0.05 \times (2 \times 1 - 1) = 1 + 0.05 \times 1 = 1.05$
$y_2 = y_1 + 0.05 \times (2 \times 1.05 - 1) = 1.05 + 0.05 \times (1.1)$
$= 1.105$
$y_3 = y_2 + 0.05 \times (2 \times y_2 - 1) = 1.1655$
$y_4 = y_3 + 0.05 \times (2 \times y_3 - 1) = 1.2321$

$h = 0.01$
use matlab w/ the following commands:

```
y(1)=1
for i=1:20
  y(i+1)=y(i)+ 0.01 x (y(i)x2-1)
end
```

y(21) is the approximate value for y at $t = 0.2$ w/ $h = 0.01$ & $y(1) = 1$.

y(21) is computed as 1.2459

$\Delta y_{h=0.1} = 1.2459 - 1.22 = 0.0259$
$\Delta y_{h=0.05} = 1.2459 - 1.2321 = 0.0138$
$\Delta y_{h=0.01} = 1.2459 - 1.2429 = 0.0029$

Problem 2: (a) $ay' + by = 0, \quad a \neq 0, \quad y(0) = 1$

$ar + b = 0, \quad r = -\frac{b}{a}, \quad y(t) = c_1 e^{-\frac{b}{a}t}, \quad c_1 = y(0) = 1$

$\boxed{y = e^{-\frac{b}{a}t}}$

(b) $y'' + 8y' - 9y = 0, \quad y(1) = 0$
$y'(1) = 1$

$r^2 + 8r - 9 = 0,$

$(r+9)(r-1) = 0$

$r = -9, 1$

$y = c_1 e^{-9t} + c_2 e^{t} \qquad y(1) = 0 = c_1 e^{-9} + c_2 e^{1}$

$y' = -9c_1 e^{-9t} + c_2 e^{t} \qquad y'(1) = 1 = -9c_1 e^{-9} + c_2 e^{1}$

$-1 = 10 c_1 e^{-9}, \quad \boxed{c_1 = \frac{-1}{10 e^{-9}} = -\frac{e^9}{10}}$

$c_1 \cdot e^{-9} + c_2 e^{1} = 0$

$-\frac{1}{10} + c_2 e = 0$

$\boxed{c_2 = -\frac{1}{10e}}$

Problem 3: $2y'' + 3y' - 2y = 0 \qquad y(0) = 1, \; y'(0) = -\beta \qquad \beta > 0$

(a) $2r^2 + 3r - 2 = 0, \quad (2r-1)(r+2) = 0, \quad r = \frac{1}{2}, -2$

$y = c_1 e^{-2t} + c_2 e^{\frac{1}{2}t} \qquad y(0) = 1 = c_1 + c_2 \quad$ — ①

$y' = -2c_1 e^{-2t} + \frac{c_2}{2} e^{\frac{t}{2}} \qquad y'(0) = -\beta = -2c_1 + \frac{c_2}{2} \quad$ — ②

① − ② × 2, $\quad 1 + 2\beta = 5c_1, \quad c_1 = \frac{1+2\beta}{5}$

$c_2 = 1 - c_1 = \frac{5-1-2\beta}{5} = \frac{4-2\beta}{5}$

$y = \frac{1+2\beta}{5} e^{-2t} + \frac{4-2\beta}{5} e^{t/2}$

(b) $\beta = 1, \quad y = \frac{3}{5} e^{-2t} + \frac{2}{5} e^{t/2}$

$y' = -2 \cdot \frac{1+2\beta}{5} e^{-2t} + \frac{1}{2} \frac{4-2\beta}{5} e^{t/2}$

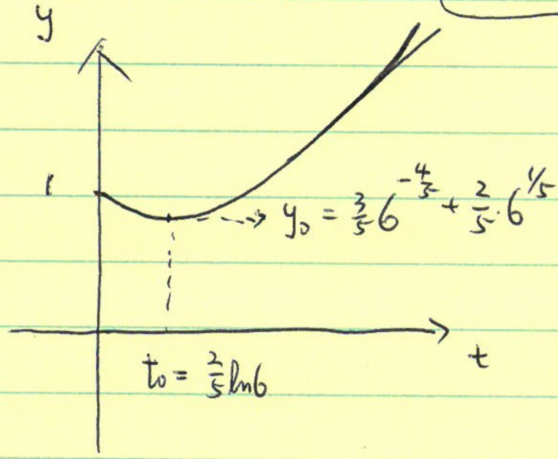
$\beta = 1,\quad y' = -\frac{6}{5}e^{-2t} + \frac{1}{5}e^{t/2} = 0$

$-\frac{6}{5}e^{-2t} + \frac{1}{5}e^{t/2} = 0,\qquad \frac{e^{-2t}}{e^{t/2}} = \frac{1/5}{6/5} = \frac{1}{6}$

$e^{5t/2} = 6 \qquad \frac{5t}{2} = \ln 6,\quad t = \frac{2}{5}\ln 6$

$\boxed{t_0 = \frac{2}{5}\ln 6},\quad y_0 = y\left(\frac{2}{5}\ln 6\right) = \frac{3}{5}e^{-\frac{4}{5}\ln 6} + \frac{2}{5}e^{\frac{1}{5}\ln 6}$

$\boxed{y_0 = \frac{3}{5}6^{-\frac{4}{5}} + \frac{2}{5}6^{\frac{1}{5}}}$



(c) $\quad y' = -2\cdot\frac{1+2\beta}{5}e^{-2t} + \frac{1}{2}\,\frac{4-2\beta}{5}e^{t/2}$

set $y' = 0$ and solve for t,

$-\frac{2}{5}(1+2\beta)e^{-2t} = -\frac{1}{5}(2-\beta)e^{t/2}$

$$\frac{2(1+2\beta)}{2-\beta} = e^{\frac{5t}{2}}$$

always a solution as long as $\frac{2(1+2\beta)}{2-\beta} > 0$, $\quad 1+2\beta > 0$ because $\beta > 0$

$2-\beta > 0,\ \boxed{2 > \beta}$

smallest value of β for which the solution has no minimum point: $\boxed{\beta = 2}$