Solutions for HW weeks 6 & 7

Problem 1: $y''+y=t(1+\sin t), \quad y(0)=0, \; y'(0)=0$

$y''+y=0, \quad r^2+1=0, \quad r=\pm i, \quad y_1=\cos t, \; y_2=\sin t$

$y_c = C_1\cos t + C_2 \sin t$

$Y = At+B + t(Ct+D)\cos t + t(Et+F)\sin t$

$Y' = A + (2Ct+D)\cos t + t(Ct+D)(-\sin t)$
$\quad + (2Et+F)\sin t + t(Et+F)\cos t$

$Y'' = 2C\cos t + (2Ct+D)(-\sin t) + (2Ct+D)(-\sin t)$
$\quad + t(Ct+D)(-\cos t)$
$\quad + 2E\cdot\sin t + (2Et+F)\cos t + (2Et+F)\cos t$
$\quad + t(Et+F)(-\sin t)$

$Y''+Y = At+B + 2C\cos t + 2(2Ct+D)(-\sin t) + 2(2Et+F)\cos t$
$\quad + 2E\cdot\sin t = t + t\sin t$

$A=1, \; B=0,$

$t\sin t: \quad 4C = 1, \quad C=\frac{1}{4}$

$t\cos t: \quad 4E = 0, \quad E=0$

$\sin t: \quad -2D+2E = 0, \quad D=0$

$\cos t: \quad 2C+2F = 0, \quad F=-C=-\frac{1}{4}$

$y = C_1\cos t + C_2\sin t + t + \frac{t^2}{4}\cos t - \frac{t}{4}\sin t$

$y' = -C_1\sin t + C_2\cos t + 1 + \frac{t}{2}\cos t + \frac{t^2}{4}(-\sin t) - \frac{1}{4}\sin t - \frac{t}{4}\cos t$

$y(0)=0 = C_1$

$y'(0)=0 = C_2+1, \quad C_2=-1$

$\boxed{y(t) = -\sin t + t + \frac{t^2}{4}\cos t - \frac{t}{4}\sin t}$

Problem 2 : $x^2y'' - 3xy' + 4y = x^2\ln x, \quad x > 0,$

$y_1 = x^2, \; y_1' = 2x, \; y_1'' = 2$

$2x^2 - 3x \cdot 2x + 4 \cdot x^2 = 6x^2 - 6x^2 = 0 \implies y_1$ is a soln. to the homogeneous eq.

$y_2 = x^2\ln x, \quad y_2' = 2x\ln x + x, \quad y_2'' = 2\ln x + 2 + 1 = 2\ln x + 3$

$x^2(2\ln x + 3) - 3x(2x\ln x + x) + 4x^2\ln x$

$= 0 \implies y_2$ is a solution to the homogeneous eq.

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix}$$

$$= 2x^3\ln x + x^3 - 2x^3\ln x = x^3$$

$$Y = \left(\int \frac{-x^2\ln x \cdot \ln x}{x^3}\,dx\right)x^2 + \left(\int \frac{x^2\ln x}{x^3}\,dx\right)x^2\ln x$$

$$= \left(\int -\frac{(\ln x)^2}{x}\,dx\right)x^2 + \left(\int \frac{\ln x}{x}\,dx\right)x^2\ln x$$

$$\int \frac{(\ln x)^2}{x}\,dx \xrightarrow{u=\ln x} \int u^2\,du = \frac{u^3}{3} = \frac{(\ln x)^3}{3}$$

$$\int \frac{\ln x}{x}\,dx \xrightarrow{u=\ln x} \int u\,du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

$$Y = -\frac{(\ln x)^3}{3} \cdot x^2 + x^2 \cdot \ln x \cdot \frac{(\ln x)^2}{2}$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right)x^2(\ln x)^3 = \frac{x^2(\ln x)^3}{6}$$

$$y = c_1x^2 + c_2x^2\ln x + \frac{x^2}{6}(\ln x)^3$$

$$y' = 2c_1x^2 + 2c_2x\ln x + c_2x + \frac{x}{3}(\ln x)^3 + \frac{x^2}{2} \cdot (\ln x)^2 \cdot \frac{1}{x}$$

P.03

$$y(1) = 0 = c_1$$

$$y'(1) = 0 = 2c_1 + c_2$$

$$c_1 = 0 = c_2 \Rightarrow \boxed{y = \frac{x^2}{6}(\ln x)^3}$$

Problem 3 $\quad u'' + \frac{1}{4}u' + 2u = 0, \qquad u(0) = 0, \quad u'(0) = 2$

$$r^2 + \frac{1}{4}r + 2 = 0$$

$$\left(r + \frac{1}{8}\right)^2 + 2 - \left(\frac{1}{8}\right)^2 = 0$$

$$\left(r + \frac{1}{8}\right)^2 + \frac{128 - 1}{64} = 0$$

$$r = -\frac{1}{8} + \frac{\sqrt{127}\,i}{8}$$

$$u_1 = e^{-\frac{t}{8}} \cdot \cos\frac{\sqrt{127}}{8}t, \quad u_2 = e^{-\frac{t}{8}} \cdot \sin\frac{\sqrt{127}}{8}t$$

$$u = c_1 e^{-\frac{t}{8}} \cdot \cos\frac{\sqrt{127}}{8}t + c_2 e^{-\frac{t}{8}} \sin\frac{\sqrt{127}}{8}t$$

$$= e^{-\frac{t}{8}} \left( c_1 \cos\frac{\sqrt{127}}{8}t + c_2 \cdot \sin\frac{\sqrt{127}}{8}t \right)$$

$$u' = -\frac{1}{8} e^{-\frac{t}{8}} \left( c_1 \cos\frac{\sqrt{127}}{8}t + c_2 \sin\frac{\sqrt{127}}{8}t \right)$$

$$+ e^{-\frac{t}{8}} \cdot \frac{\sqrt{127}}{8} \left( -c_1 \sin\frac{\sqrt{127}}{8}t + c_2 \cos\frac{\sqrt{127}}{8}t \right)$$

$$u(0) = 0 = c_1$$

$$u'(0) = -\frac{1}{8}(\cdot 0) + \frac{\sqrt{127}}{8}c_2 = 2, \qquad c_2 = \frac{16}{\sqrt{127}}$$

$$u = \frac{16}{\sqrt{127}} e^{-\frac{t}{8}} \sin\frac{\sqrt{127}}{8}t$$

$$\frac{16}{\sqrt{127}} e^{-\frac{T}{8}} \le \frac{1}{2} \cdot \frac{16}{\sqrt{127}}$$

$$e^{-\frac{T}{8}} \le \frac{1}{2}$$

$$\frac{T}{8} \ge \ln 2$$

$$T \ge \ln 2 + \ln 8 = \underline{\underline{4\ln 2}}$$

Problem 17 $u'' + \frac{1}{4}u' + 2u = 2\cos\omega t, \quad u(0) = 0, \quad u'(0) = 2$

$r^2 + \frac{1}{4}r + 2 = 0, \quad \left(r + \frac{1}{8}\right)^2 + 2 - \frac{1}{64} = 0, \quad \left(r + \frac{1}{8}\right)^2 + \frac{127}{64} = 0$

$r = -\frac{1}{8} \pm \frac{\sqrt{127}}{8} i$

$u_c = e^{-\frac{t}{8}} \left(c_1 \cos\frac{\sqrt{127}}{8}t + c_2 \sin\frac{\sqrt{127}}{8}t\right)$

$U = A\cos\omega t + B\sin\omega t$

$U' = -\omega A \sin\omega t + \omega B \cos\omega t$

$U'' = -\omega^2 A \cos\omega t - \omega^2 B \sin\omega t$

$-\omega^2(A\cos\omega t + B\sin\omega t) + \frac{1}{4}(-\omega A \sin\omega t + \omega B \cos\omega t) + 2(A\cos\omega t + B\sin\omega t)$

$= 2\cos\omega t$

$\cos\omega t$: $-\omega^2 A + \frac{\omega B}{4} + 2A = 2$

$\sin\omega t$: $-\omega^2 B - \frac{\omega A}{4} + 2B = 0 \qquad (\omega^2 - 2)B = -\frac{\omega}{4}A, \quad B = -\frac{\omega}{4(\omega^2-2)}A$

$(2-\omega^2)A + \frac{\omega}{4}\cdot\left(-\frac{\omega}{4(\omega^2-2)}\right)A = 2$

$(2-\omega^2)A + \frac{\omega^2 A}{16(2-\omega^2)} = 2$

$\frac{16(2-\omega^2)^2 A + \omega^2 A}{16(2-\omega^2)} = 2, \qquad A = \frac{32(2-\omega^2)}{\omega^2 + 16(2-\omega^2)^2}$

$B = -\frac{\omega}{4(\omega^2-2)} \cdot \frac{32(2-\omega^2)}{\omega^2 + 16(2-\omega^2)^2}$

$B = \frac{8\omega}{\omega^2 + 16(2-\omega^2)^2}$

$u = \left(c_1 \cos\frac{\sqrt{127}}{8}t + c_2 \sin\frac{\sqrt{127}}{8}t\right)e^{-\frac{t}{8}} + \frac{32(2-\omega^2)}{\omega^2 + 16(2-\omega^2)^2}\cos\omega t$

$+ \frac{8\omega}{\omega^2 + 16(2-\omega^2)^2}\sin\omega t$

(a) Steady state part of the solution is $U$

$U = \frac{8}{\omega^2 + 16(2-\omega^2)^2}\left(\omega \sin\omega t + 4(2-\omega^2)\cos\omega t\right)$

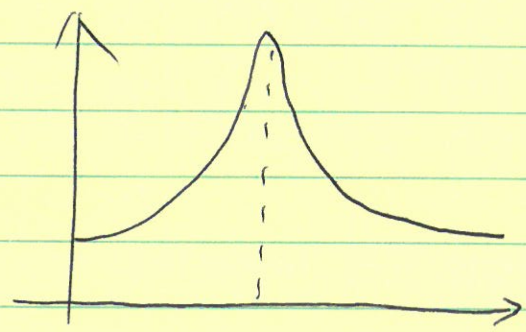
P.05

$$U = \frac{8}{\omega^2+16(2-\omega^2)^2} \cdot \sqrt{\omega^2+16(2-\omega^2)^2} \cdot \left( \cos\omega t \cdot \frac{4(2-\omega^2)}{\sqrt{\omega^2+16(2-\omega^2)^2}} + \sin\omega t \cdot \frac{\omega}{\sqrt{\omega^2+16(2-\omega^2)^2}} \right)$$

$$= \frac{8}{\cancel{8}\cdot\sqrt{\omega^2+16(2-\omega^2)^2}} \cdot \left( \cos\omega t \cdot \cos\theta + \sin\omega t \sin\theta \right)$$

$$= 8/\sqrt{\omega^2+16(2-\omega^2)^2} \cdot \cos(\omega t - \theta), \quad \theta = \tan^{-1}\left(\frac{\omega}{4(2-\omega^2)}\right)$$

(c)




(d) $R = \frac{8}{\sqrt{\omega^2+16(2-\omega^2)^2}}$ $\quad \frac{dR}{d\omega} = 0$, at $\omega = \omega_{max} = \boxed{\frac{64}{\sqrt{127}}}$

Problem 18: $u'' + u = 3\cos\omega t, \quad u(0) = 0, \quad u'(0) = 0$

$r^2 + r^0 = 0. \quad r = \pm i, \quad u_1 = \cos t, \quad u_2 = \sin t$

$U = A\cos\omega t + B\sin\omega t$

$U' = -\omega A\sin\omega t + \omega B\cos\omega t$

$U'' = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$

$U'' + U = (1-\omega^2)A\cos\omega t + (1-\omega^2)B\sin\omega t = 3\cos\omega t$

$\omega \neq 1, \quad \begin{cases} A = \frac{3}{1-\omega^2}, \\ B = 0 \end{cases}$

$u = c_1\cos t + c_2\sin t + \frac{3}{1-\omega^2}\cos\omega t$

$u' = -c_1\sin t + c_2\cos t - \frac{3\omega}{1-\omega^2}\sin\omega t$

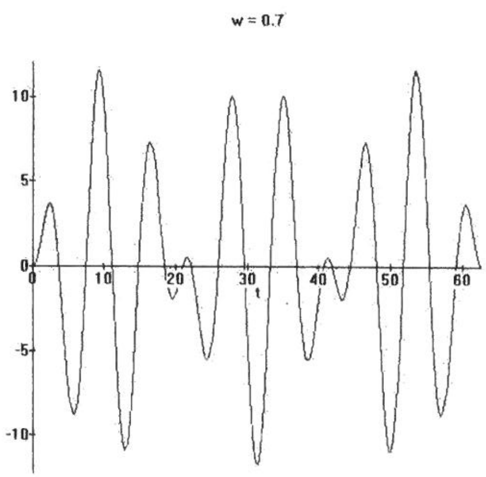
$u(0) = 0 \quad c_1 + \frac{3}{1-\omega^2} = 0 \quad c_1 = -\frac{3}{1-\omega^2}$

$u'(0) = 0 \quad c_2 = 0$

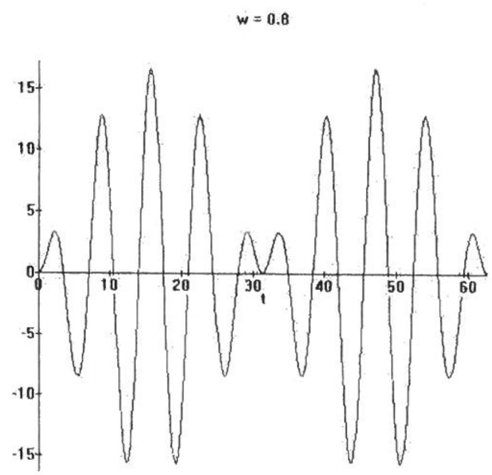
$u = -\frac{3}{1-\omega^2}\cos t + \frac{3}{1-\omega^2}\cos\omega t = \frac{3}{1-\omega^2}\cdot(\cos\omega t - \cos t)$

$u = \frac{6}{1-\omega^2}\sin\left(\frac{1-\omega}{2}t\right)\cdot\sin\left(\frac{1+\omega}{2}t\right)$

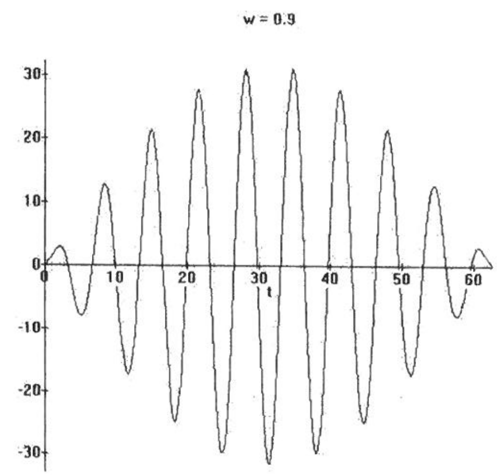
(b)




(a) $\omega = 0.7$




(b) $\omega = 0.8$




(c) $\omega = 0.9$