Solutions for HW week 9 & week 10

Problem 11 page 286

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

(a)

$P(x) = 1-x^2$

$Q(x) = -2x$

$R(x) = \alpha(\alpha+1)$

$$p_0 = \lim_{x\to 1} \frac{(x-1)\,Q(x)}{P(x)} = \lim_{x\to 1} \frac{(x-1)\cdot(-2x)}{1-x^2} = \lim_{x\to 1} \frac{2x}{x+1} = 1$$

$$q_0 = \lim_{x\to 1} \frac{(x-1)^2\cdot \alpha(\alpha+1)}{1-x^2} = \lim_{x\to 1} \frac{(x-1)(x-1)\alpha(\alpha+1)}{(1+x)(1-x)} = 0$$

indicial equation: $x^2y'' + xy' + 0y = 0$

$r(r-1) + r = 0 \quad r^2 - r + r = 0, \quad r = 0, 0$

(b) $y = \sum_{n=0}^{\infty} a_n (x-1)^n = \sum_{n=0}^{\infty} a_n t^n, \quad t = x-1, \quad x+1 = x-1+2 = t+2$

$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

$-t\cdot(t+2)y'' - 2(t+1)y' + \alpha(\alpha+1)y = 0 \Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}$

$y' = \sum_{n=1}^{\infty} a_n n t^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$

$$-t(t+2)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 2(t+1)\sum_{n=1}^{\infty} n a_n t^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n t^n = 0,$$

$$-\sum_{n=2}^{\infty} n(n-1)a_n t^n - 2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} - 2\sum_{n=1}^{\infty} n a_n t^n - 2\sum_{n=1}^{\infty} n a_n t^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n t^n = 0,$$

t^0: $-2a_1 + \alpha(\alpha+1)a_0 = 0, \quad a_1 = \frac{\alpha(\alpha+1)}{2}a_0$

t^1: $-2\cdot 2\cdot 1\cdot a_2 - 2a_1 - 2\cdot 2\cdot a_2 + \alpha(\alpha+1)a_1 = 0$

$-8a_2 = (2 - \alpha(\alpha+1))a_1,$

$a_2 = \frac{\alpha(\alpha+1) - 2}{8} a_1$

t^2: $-2\cdot 1\cdot a_2 - 2\cdot 3\cdot 2\cdot a_3 - 2\cdot 2\cdot a_2 - 2\cdot 3\cdot a_3 + \alpha(\alpha+1)a_2 = 0$

$$-18a_3 + (\alpha(\alpha+1)-6)a_2 = 0, \qquad a_3 = \frac{\alpha(\alpha+1)-6}{18}a_2$$

t^n: $\quad -n(n-1)a_n - 2(n+1)\cdot n\cdot a_{n+1} - 2\cdot n\cdot a_n - 2(n+1)a_{n+1} + \alpha(\alpha+1)a_n = 0$

$$a_{n+1} = -\frac{n^2+n-\alpha(\alpha+1)}{2(n+1)^2}a_n$$

Problem 2: $\quad y' + ay = e^{\lambda t}, \quad y(0) = c, \quad a \neq 0$

$$\mathcal{L}[y'+ay] = sY - y(0) + aY = \frac{1}{s-\lambda}$$

$$(s+a)Y = c + \frac{1}{s-\lambda}$$

$$Y = \frac{c}{s+a} + \frac{1}{(s+a)(s-\lambda)}$$

if $a \neq -\lambda$ $\quad Y = \frac{c}{s+a} + \frac{A}{s+a} + \frac{B}{s-\lambda}, \quad A(s-\lambda)+B(s+a) = 1$

$A + B = 0 \qquad A = -B$

$-A\lambda + Ba = 1, \quad B(\lambda+a) = 1,$

$B = \frac{1}{\lambda+a}$

$$Y = \frac{c}{s+a} - \frac{1/(\lambda+a)}{s+a} + \frac{1/(\lambda+a)}{s-\lambda}$$

$$\boxed{y(t) = \mathcal{L}^{-1}[Y] = c\cdot e^{-at} - \frac{1}{\lambda+a}e^{-at} + \frac{1}{\lambda+a}e^{\lambda t}}$$

if $a = -\lambda$ $\quad Y = \frac{c}{s+a} + \frac{1}{(s+a)^2}$

$\boxed{\mathcal{L}[Y] = c\cdot e^{-at} + te^{-at}}$ from # 11 in Table 6.2.1.

Problem 3: (Problem 25 on page 333 334)

(a) $\mathcal{L}[f(ct)] = \int_0^\infty e^{-st} f(ct)\,dt = \int_0^\infty e^{-\frac{s}{c}ct} f(ct)\frac{dct}{c}$, $\quad c > 0$

$$= \frac{1}{c}\cdot F\left(\frac{s}{c}\right)$$

(b) $\mathcal{L}^{-1}[F(ks)] = ?$ $\quad$ let $k = \frac{1}{c} > 0, \quad c = \frac{1}{k} > 0$

$$kF(ks) = \mathcal{L}[f(\tfrac{t}{k})]$$

$$\mathcal{L}^{-1}[F(ks)] = \mathcal{L}^{-1}\mathcal{L}\left[\frac{1}{k}f\left(\frac{t}{k}\right)\right] = \frac{1}{k}f\left(\frac{t}{k}\right)$$

(c) $\mathcal{L}[\frac{1}{a}e^{-\frac{b}{a}t}f(\frac{t}{a})] = \int_0^\infty e^{-st}\cdot\frac{1}{a}e^{-\frac{b}{a}t}f(\frac{t}{a})\,dt$ by definition of Laplace transform

$$= \frac{1}{a}\int_0^\infty e^{-(s+\frac{b}{a})t} f(\tfrac{t}{a})\,dt \qquad \text{let } \tau = \frac{t}{a},\ d\tau = dt/a$$

$$= \int_0^\infty e^{-(as+b)\tau} f(\tau)\,d\tau$$

$$= F(as+b) \quad \text{if } F(s) \equiv \int_0^\infty e^{-st} f(t)\,dt$$

Therefore $\mathcal{L}^{-1}[F(as+b)] = \frac{1}{a}e^{-\frac{b}{a}t}f(\frac{t}{a})$ ※

Problem 4
(Problem 32 page 334)

$$f(t) = 1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t)$$

$$\mathcal{L}[f] = \mathcal{L}\left[1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t)\right]$$

$$= \frac{1}{s} + \sum_{k=1}^{2n+1} (-1)^k \cdot \frac{e^{-ks}}{s}$$

$$= \frac{1}{s}\left(1 + \sum_{k=1}^{2n+1} (-1)^k e^{-ks}\right) = \frac{1}{s}\cdot\frac{1-(-e^{-s})^{2n+2}}{1-(-e^{-s})}$$

$$= \frac{1}{s}\,\frac{1 - (-1)^{2n+2} e^{-(2n+2)s}}{1+e^{-s}}$$ ※

Problem 5
(Problem 16 p 341)

$$u'' + \frac{1}{4}u' + u = k\cdot(u_{3/2} - u_{5/2}),\quad k>0,\quad u(0) = 0 = u'(0)$$

$$\mathcal{L}[u'' + \tfrac{1}{4}u' + u] = k\,\mathcal{L}[u_{3/2} - u_{5/2}] = k\cdot\left(\frac{e^{-\frac{3s}{2}}}{s} - \frac{e^{-5s/2}}{s}\right)$$

$$s^2U + \frac{1}{4}sU + U = \frac{k}{s}\cdot(e^{-\frac{3s}{2}} - e^{-\frac{5s}{2}})$$

$$U(s^2 + \tfrac{1}{4}s + 1) = \frac{k}{s}(e^{-\frac{3s}{2}} - e^{-\frac{5s}{2}})$$

$$U = \frac{k}{s(s^2+\frac{1}{4}s+1)}(e^{-\frac{3s}{2}} - e^{-5s/2})$$

P.04

$$\frac{1}{s(s^2+\frac{1}{4}s+1)} = \frac{1}{s(s^2+\frac{1}{4}s+\frac{1}{64}+1-\frac{1}{64})} = \frac{1}{s\left((s+\frac{1}{8})^2+\left(\frac{\sqrt{63}}{8}\right)^2\right)}$$

$$= \frac{A}{s} + \frac{Bs+C}{(s+\frac{1}{8})^2+\left(\frac{\sqrt{63}}{8}\right)^2}$$

$$A(s^2+\tfrac{1}{4}s+1) + Bs^2 + Cs = 1$$

$A+B=0 \quad B=-1$

$\frac{A}{4}+C=0 \quad C=-\frac{1}{4}$

$A=1$

$$\frac{1}{s(s^2+\frac{1}{4}s+1)} = \frac{1}{s} + \frac{-s-\frac{1}{4}}{(s+\frac{1}{8})^2+\left(\frac{\sqrt{63}}{8}\right)^2} = \frac{1}{s} - \frac{s+\frac{1}{8}+\frac{1}{8}}{(s+\frac{1}{8})^2+\left(\frac{\sqrt{63}}{8}\right)^2}$$

$$= \frac{1}{s} - \frac{s+\frac{1}{8}}{(s+\frac{1}{8})^2+\left(\frac{\sqrt{63}}{8}\right)^2} - \frac{1}{\sqrt{63}}\frac{\sqrt{63}/8}{(s+\frac{1}{8})^2+\left(\frac{\sqrt{63}}{8}\right)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+\frac{1}{4}s+1)}\right] = 1 - e^{-\frac{t}{8}}\cos\frac{\sqrt{63}}{8}t - \frac{1}{\sqrt{63}}e^{-\frac{t}{8}}\sin\frac{\sqrt{63}}{8}t \equiv h(t)$$

$$\therefore\ u(t) = k\cdot u_{3/2}\,h(t-3/2) - k\,u_{5/2}\,h(t-5/2)$$

By plotting the solutions in Matlab, the solutions appear to ~~have~~ be able to reach a value of $y=2$ as long as $\boxed{k>2.51}$

By plotting the solution w/ $k=2$, numerical calculation shows that $\boxed{|u(t)|<0.1 \text{ for } t>25.6733.}$

Problem 6 (Problem 14 page 348)

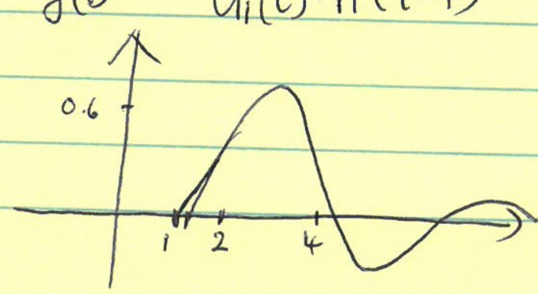
$$y''+\gamma y'+y=\delta(t-1) \qquad y(0)=0,\ y'(0)=0$$

(a) $\gamma=\frac{1}{2}$,

$$(s^2+\tfrac{1}{2}s+1)Y=e^{-s}, \quad Y=\frac{e^{-s}}{s^2+\frac{1}{2}s+1}=\frac{e^{-s}}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right] = \frac{4}{\sqrt{15}}\cdot e^{-\frac{t}{4}}\sin\frac{\sqrt{15}}{4}t \equiv h(t)$$

$$y(t) = u_1(t)\cdot h(t-1)$$



$$y(t) = \begin{cases} 0 & t \le 1 \\ h(t-1) & t > 1 \end{cases}$$

$$h(t-1) = \frac{4}{\sqrt{15}} e^{-\frac{(t-1)}{4}} \sin\frac{\sqrt{15}}{4}(t-1)$$

for $t > 1$

$$y' = \frac{4}{\sqrt{15}} \cdot \left(-\frac{1}{4}\right) e^{-\frac{(t-1)}{4}} \cdot \sin\frac{\sqrt{15}}{4}(t-1) + \frac{4}{\sqrt{15}} \cdot e^{-\frac{(t-1)}{4}} \cdot \frac{\sqrt{15}}{4} \cos\frac{\sqrt{15}}{4}(t-1) = 0$$

$$-\frac{1}{4} \sin\frac{\sqrt{15}}{4}(t-1) + \frac{\sqrt{15}}{4} \cos\frac{\sqrt{15}}{4}(t-1) = 0$$

$$\frac{\sin\frac{\sqrt{15}}{4}(t-1)}{\cos\frac{\sqrt{15}}{4}(t-1)} = \tan\frac{\sqrt{15}}{4}(t-1) = \frac{\sqrt{15}/4}{1/4} = \sqrt{15}$$

$$\frac{\sqrt{15}}{4}(t-1) = \tan^{-1}\sqrt{15}, \quad t = 1 + \frac{4}{\sqrt{15}} \tan^{-1}\sqrt{15},$$

$$\boxed{\begin{array}{l} t_1 \sim 2.3613 \\ y(2.3613) \sim 0.71153 \end{array}}$$

(c) $\gamma = 1/4$, $\quad Y = \dfrac{e^{-s}}{s^2 + \frac{s}{4} + 1}$, $\quad y(t) = \dfrac{8}{3\sqrt{7}} e^{-(t-1)/8} \sin\frac{3\sqrt{7}}{8}(t-1)\, u_1(t)$

$$\boxed{t_1 \sim 2.4569, \quad y(t_1) \sim 0.8335}$$

(d) for any γ,

$$Y = \frac{e^{-s}}{s^2 + \gamma s + 1} = \frac{e^{-s}}{\left(s + \frac{\gamma}{2}\right)^2 + \left(1 - \frac{\gamma^2}{4}\right)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{\gamma}{2}\right)^2 + 1 - \frac{\gamma^2}{4}}\right] = \frac{1}{\sqrt{1 - \frac{\gamma^2}{4}}} e^{-\frac{\gamma}{2}t} \cdot \sin\sqrt{1 - \frac{\gamma^2}{4}}\, t \equiv h(t)$$

$$y(t) = u_1(t)\, h(t-1)$$

for $t > 1$ $\quad y(t) = h(t-1)$, $\quad y' = 0$, get

$$\tan\left[\sqrt{1 - \gamma^2/4} \cdot (t-1)\right] = \frac{1}{\gamma}\sqrt{4 - \gamma^2}$$

$$\frac{\tan\sqrt{1-\gamma^2/4}\,(t-1)}{\sqrt{1-\gamma^2/4}} = \frac{4}{2\gamma} = \frac{2}{\gamma}$$

$\gamma \to 0 \quad \tan(t-1) \to \infty, \quad t \Rightarrow 1 + \frac{\pi}{2}$

when $\gamma \to 0$
$y \to u_1 \sin(t-1)$

Problem 7

(Problem 15, P 348)

$$y'' + \gamma y' + y = k\,\delta(t-1), \qquad y(0) = y'(0) = 0$$

(a)

$$(s^2 + \gamma s + 1)\, Y = k \cdot e^{-s}$$

$$Y = \frac{k e^{-s}}{s^2 + \gamma s + 1} = \frac{k e^{-s}}{\left(s + \frac{\gamma}{2}\right)^2 + 1 - \frac{\gamma^2}{4}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{\gamma}{2}\right)^2 + 1 - \frac{\gamma^2}{4}}\right] = \frac{1}{\sqrt{1-\frac{\gamma^2}{4}}}\, e^{-\frac{\gamma}{2}t} \cdot \sin\left(\sqrt{1-\frac{\gamma^2}{4}}\, t\right) \equiv h(t)$$

$$y(t) = k \cdot h(t-1) \cdot u_1(t)$$

Similar to problem 6 (problem 14 in page 348)

(b) $t_1 \sim 2.4569$ when $y(t_1) \sim 0.8335k$ is the maximum

$y = 2$ when $k_1 = 2/0.8335 \sim 2.3995$

(c) as $\gamma \to 0$, $y \to k \sin(t-1)\, u_1$ and $t_1 - 1 \to \pi/2$

$\Rightarrow$ Requiring that the peak value remains at $y = 2$, the limiting value of k is $k_1 = 2$.