Math 222, Spring 2017.

Present your work in an organized fashion. Make sure that your work is algebraically correct and logically sound. Show all your work. No calculator, notes, or books.

In-class Quiz, on 04/07/2017 for M222-008

1. Find the solution of the initial value problem

$$y'' - y = f(t);\ y(0) = 0,\ y'(0) = 1;\ f(t) = \begin{cases} e^t, & 0 \le t < 3\pi \\ 0, & 3\pi \le t < \infty \end{cases}$$

2. Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t-5);\ y(0) = 0,\ y(0) = -1.$$

3. Use the convolution theorem to express the inverse Laplace transform of the given function in a convolution integral

(a)

$$F(s) = \frac{s^2}{(s^2+1)(4s^2+1)},$$

(b)

$$F(s) = \frac{G(s)}{s^6},$$

(c)

$$F(s) = \frac{1}{(s+1)(s^2-4)}.$$

Solutions for QUIZ 0407, Math 222-008, 2017

Problem 1. $y''-y=f(t)=\begin{cases} e^t & 0\le t<3\pi \\ 0 & 3\pi\le t\end{cases}$, $y(0)=0$, $y'(0)=1$

$$f(t)=e^t-e^t\cdot U_{3\pi}$$

$$\mathcal{L}[y''-y]=\mathcal{L}[e^t-U_{3\pi}e^t]=\frac{1}{s-1}-\mathcal{L}[U_{3\pi}e^{t-3\pi}\cdot e^{3\pi}]$$

$$s^2Y-y'(0)-sy(0)-Y=\frac{1}{s-1}-e^{3\pi}\frac{e^{-3\pi s}}{s-1}$$

$$(s^2-1)Y=1+\frac{1}{s-1}-e^{-3\pi}\frac{e^{-3\pi s}}{s-1}$$

$$Y=\frac{1}{s^2-1}+\frac{1}{(s^2-1)(s-1)}-e^{3\pi}\cdot\frac{e^{-3\pi s}}{(s-1)(s^2-1)}$$

$$y=\mathcal{L}^{-1}[Y]=\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right]+\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s-1)^2}-e^{3\pi}\cdot\frac{e^{-3\pi s}}{(s+1)(s-1)^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right]=\sinh t$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s-1)^2}\right]=\mathcal{L}^{-1}\left[\frac{A}{s+1}+\frac{Bs+C}{(s-1)^2}\right]$$

$$A(s-1)^2+(Bs+C)(s+1)=1$$

s^2: $A+B=0$ $\qquad A=-B$

s^1: $-2A+C+B=0$ $\qquad 3B+C=0$

s^0: $A+C=1$ $\qquad -B+C=1$ $\qquad 4B=-1,\ B=-\frac{1}{4}$

$A=\frac{1}{4},\ C=\frac{3}{4}$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s-1)^2}\right]=\mathcal{L}^{-1}\left[\frac{\frac{1}{4}}{s+1}+\frac{-\frac{1}{4}s+\frac{3}{4}}{(s-1)^2}\right]=\frac{1}{4}e^{-t}-\frac{1}{4}\mathcal{L}^{-1}\left[\frac{s-3}{(s-1)^2}\right]$$

$$=\frac{1}{4}e^{-t}-\frac{1}{4}\mathcal{L}^{-1}\left[\frac{s-1-2}{(s-1)^2}\right]=\frac{1}{4}e^{-t}-\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-1}-\frac{2}{(s-1)^2}\right]$$

$$=\frac{1}{4}e^{-t}-\frac{1}{4}e^{t}+\frac{1}{2}te^{t}\equiv h(t)$$

$$y(t)=\sinh t+h(t)-e^{3\pi}\cdot U_{3\pi}(t)\,h(t-3\pi)$$

Problem 2 $2y''+y'+2y=\delta(t-5)$, $y(0)=0$, $y'(0)=-1$

$$2(s^2Y-y'(0)-sy(0))+sY-y(0)+2Y=e^{-5s}$$

$$(2s^2+s+2)Y+2=e^{-5s}$$

$$Y=\frac{-2}{2s^2+s+2}+\frac{e^{-5s}}{2s^2+s+2}$$

$$Y = -\frac{2}{2(s^2+\frac{1}{2}s+(\frac{1}{4})^2)+2-\frac{1}{8}} + \frac{e^{-5s}}{2s^2+s+2}$$

$$Y = -\frac{1}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2} + \frac{e^{-5s}}{2\left((s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2\right)}$$

$$y(t) = \mathcal{L}^{-1}[Y] = -\frac{e^{-\frac{t}{4}} \sin\frac{\sqrt{15}}{4}t}{\frac{\sqrt{15}}{4}} + \frac{1}{2} \cdot u_5(t) \cdot \frac{e^{-\frac{1}{4}(t-5)} \sin\frac{\sqrt{15}}{4}(t-5)}{\frac{\sqrt{15}}{4}}$$

Problem 3: (a) $F(s) = \frac{s^2}{(s^2+1)(4s^2+1)}$, $\mathcal{L}^{-1}\left[\frac{s}{s^2+1} \cdot \frac{s}{4s^2+1}\right] = \int_0^t f(t-\tau)g(\tau)d\tau$

where $f(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$

$g(t) = \mathcal{L}^{-1}\left[\frac{s}{4s^2+1}\right] = \frac{1}{4}\cos\frac{t}{2}$

$$\therefore \mathcal{L}^{-1}[F] = \int_0^t \cos(t-\tau)\,\frac{1}{4}\cdot\cos\frac{\tau}{2}\,d\tau$$

(b) $F(s) = \frac{G(s)}{s^2}$, $\mathcal{L}^{-1}[F] = \int_0^t (t-\tau)g(\tau)d\tau$

where $\mathcal{L}^{-1}[G] = g(t)$

(c) $F(s) = \frac{1}{(s+1)(s^2-4)}$, $\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2-4)}\right] = \int_0^t e^{-(t-\tau)}\,\frac{1}{2}\sinh 2\tau\,d\tau$