Math 222, Fall 2016.

Present your work in an organized fashion. Make sure that your work is algebraically correct and logically sound. Show all your work. No calculator, notes, or books.

Take-home Quiz 11/11/2016 M222-001, Due in class on 11/14/2016

1. Find the inverse Laplace transform of the given functions.

$$(a) F(s) = \frac{3}{s^2+4}, \quad (b) F(s) = \frac{2s-3}{s^2-4}, \quad (c) F(s) = \frac{1-2s}{s^2+2s+10}.$$

2. Use Laplace transform to find the solution of the IVP:

$$y^{(4)} - y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0.$$

3. Use Laplace transform to find the solution of the IVP:

$$y'' + 4y' = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < \infty; \end{cases} \quad y(0) = 0, \quad y'(0) = 0.$$

QUIZ 11/11/2016 Math 222 Fall 2016

1. (a) $F(s) = \frac{3}{s^2+4}$,

$F(s) = \frac{3}{2} \cdot \frac{2}{s^2+2^2}$, $\mathcal{L}^{-1}[F] = \frac{3}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right] = \frac{3}{2}\cdot \sin 2t$ from #5 in table 6.2.1

(b) $F(s) = \frac{2s-3}{s^2-4} = \frac{2s-3}{(s+2)(s-2)} = \frac{A}{s+2} + \frac{B}{s-2}$

$A(s-2) + B(s+2) = 2s-3$,

$s=2$, $4B = 1$, $B = \frac{1}{4}$

$s=-2$, $-4A = -7$, $A = \frac{7}{4}$

$F(s) = \frac{7}{4}\frac{1}{s+2} + \frac{1}{4}\frac{1}{s-2}$, $\mathcal{L}^{-1}[F] = \frac{7}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$

$= \frac{7}{4}e^{-2t} + \frac{1}{4}e^{2t}$

(c) $F(s) = \frac{1-2s}{s^2+2s+10} = \frac{1-2s}{(s+1)^2+3^2} = \frac{-2(s+1)+3}{(s+1)^2+3^2}$

$\mathcal{L}^{-1}[F] = \mathcal{L}^{-1}\left[\frac{-2(s+1)+3}{(s+1)^2+3^2}\right] = \mathcal{L}^{-1}\left[\frac{-2(s+1)}{(s+1)^2+3^2} + \frac{3}{(s+1)^2+3^2}\right]$

$= -2\cdot e^{-t}\cdot \cos 3t + e^{-t}\sin 3t$

2. $y^{(4)} - y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y^{(3)}(0) = 0$

$\mathcal{L}[y^{(4)} - y] = 0$

$s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y^{(3)}(0) - Y = 0$

$(s^4-1)Y = s^3 - 2s$, $Y = \frac{s(s^2-2)}{s^4-1} = \frac{s(s^2-1-1)}{(s^2+1)(s^2-1)} = \frac{s(s^2-1)-s}{(s^2+1)(s^2-1)}$

$Y = \frac{s}{s^2+1} - \frac{s}{(s^2+1)(s^2-1)} = \frac{s}{s^2+1} - \left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1}\right)$

$(As+B)(s^2-1) + (Cs+D)(s^2+1) = s$

s^3: $A + C = 0$, s^2: $B + D = 0$, s^1: $-A + C = 1$, s^0: $-B + D = 0$

$$B=D=0, \quad A=-\frac{1}{2}, \; C=\frac{1}{2}$$

$$Y=\frac{s}{s^2+1}+\frac{\frac{1}{2}s}{s^2+1}-\frac{\frac{1}{2}s}{s^2-1}$$

$$=\frac{\frac{3}{2}s}{s^2+1}-\frac{\frac{1}{2}s}{s^2-1}=\frac{3}{2}\frac{s}{s^2+1}-\frac{1}{2}\left(\frac{\frac{1}{2}}{s+1}+\frac{\frac{1}{2}}{s-1}\right)$$

$$y(t)=\mathcal{L}^{-1}[Y]=\frac{3}{2}\cdot\cos t-\frac{1}{4}\left(e^{-t}\right)-\frac{1}{4}e^{t}$$

$$y(t)=\frac{3}{2}\cos t-\frac{1}{2}\cdot\cosh t$$

3. $$y''+4y'=\begin{cases} t & 0\le t<1 \\ 0 & 1\le t<\infty \end{cases} \qquad y(0)=0,\; y'(0)=0$$

$$\mathcal{L}[y''+4y']=\int_0^\infty e^{-st}\begin{cases} t \\ 0 \end{cases} dt=\int_0^1 e^{-st}\cdot t\,dt+0$$

$$s^2Y+4sY=\frac{e^{-st}}{-s}\cdot t\Big|_0^1-\int_0^1\frac{e^{-st}}{-s}dt$$

$$=\frac{e^{-s}}{-s}-\frac{1}{s^2}e^{-st}\Big|_0^1=\frac{e^{-s}}{-s}-\frac{1}{s^2}(e^{-s}-1)$$

$$(s^2+4s)Y=\frac{1}{s^2}-\frac{1}{s^2}e^{-s}-\frac{e^{-s}}{s}$$

$$Y=\frac{1}{s(s+4)}\cdot\left(\frac{1}{s^2}-\frac{1}{s^2}e^{-s}-\frac{e^{-s}}{s}\right)$$

$$=\frac{1}{s^3(s+4)}-\frac{1}{s^3(s+4)}\cdot e^{-s}-\frac{e^{-s}}{s^2(s+4)}$$

$$\frac{1}{s^3(s+4)}=\frac{As^2+Bs+C}{s^3}+\frac{D}{s+4}$$

$$(As^2+Bs+C)(s+4)+Ds^3=1$$

s^3: $A+D=0$ $\quad D=-\frac{1}{64}$

s^2: $4A+B=0$ $\quad A=\frac{1}{64}$

s^1: $4B+C=0$ $\quad B=-\frac{1}{16}$

s^0: $4C=1$ $\quad C=\frac{1}{4}$

$$\frac{1}{s^2(s+4)} = \frac{As+B}{s^2} + \frac{C}{s+4}$$

$$(As+B)(s+4) + Cs^2 = 1$$

s^2: $A + C = 0$ $\quad C = \frac{1}{16}$

s^1: $4A + B = 0$ $\quad A = -\frac{1}{16}$

s^0: $4B = 1$ $\quad B = \frac{1}{4}$

$$Y = \frac{\frac{1}{64}s^2 - \frac{1}{16}s + \frac{1}{4}}{s^3} + \frac{-\frac{1}{64}}{s+4} - \frac{e^{-s}}{s^3(s+4)} - \left(\frac{-\frac{1}{16}s + \frac{1}{4}}{s^2} + \frac{\frac{1}{16}}{s+4}\right)e^{-s}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3(s+4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{64}\frac{1}{s} - \frac{1}{16}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s^3}\right] + \mathcal{L}^{-1}\left[\frac{-\frac{1}{64}}{s+4}\right]$$

$$= \frac{1}{64} - \frac{1}{16}t + \frac{1}{8}t^2 - \frac{1}{64}e^{-4t} \equiv h(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+4)}\right] = \mathcal{L}^{-1}\left[\frac{-\frac{1}{16}s + \frac{1}{4}}{s^2} + \frac{\frac{1}{16}}{s+4}\right] = -\frac{1}{16} + \frac{1}{4}t + \frac{1}{16}e^{-4t} \equiv g(t)$$

$$y = \mathcal{L}^{-1}[Y] = h(t) - u_1(t)\,h(t-1) - u_1(t)\,g(t-1)$$