prob $\overset{12}{13}$, 14, $\overset{26}{,}30$

02/09/05 M614 P.1

Show that $f[x_0, x_1, \cdots, x_k] = \frac{1}{k!\,h^k} \Delta^k f_0$

$k=0$ $\quad f[x_0] = f_0$

$k=1$ $\quad f[x_0, x_1] = \frac{1}{h}\Delta^1 f_0 = \frac{1}{h}(f(x_0) + f(x_1))$

$\vdots$

$k=r$ $\quad f[x_0, x_1, \cdots, x_r] = \frac{1}{r!\,h^r}\Delta^r f_0$

$k=r+1$ $\quad f[x_0, \cdots, x_r, x_{r+1}] = \dfrac{f[x_1, \cdots, x_{r+1}] - f[x_0, \cdots, x_r]}{x_{r+1} - x_0}$

$$= \frac{\frac{1}{r!\,h^r}\Delta^r f_1 - \frac{1}{r!\,h^r}\Delta^r f_0}{x_{r+1} - x_0}$$

$$= \frac{1}{r!\,h^r}\frac{1}{(r+1)h}(\Delta^r f_1 - \Delta^r f_0)$$

$$= \frac{1}{(r+1)!\,h^{r+1}}\Delta^{r+1} f_0$$

by induction,

We prove that

$$f[x_0, x_1, \cdots, x_k] = \frac{1}{k!\,h^k}\Delta^k f_0$$ ✠

Newton's forward divided differences:

$$P_{n+1}(x) = f(x_0) + (x-x_0) f[x_0, x_1] + \cdots$$
$$+ (x-x_0) \cdots (x-x_{n-1}) f[x_0, \cdots, x_n]$$
$$+ (x-x_0) \cdots (x-x_n) f[x_0, \cdots, x_n, t]$$

$$= P_n(x) + (x-x_0) \cdots (x-x_n) f[x_0, x_1, \cdots, x_n, t]$$

using the formula

$$f[x_0, x_1, \cdots, x_k] = \frac{1}{k!\,h^k} \Delta^k f_0 ,$$

we can show that

$$P_n(x) = \sum_{j=0}^{n} \binom{\mu}{j} \Delta^j f_0 , \quad \mu \equiv \frac{x-x_0}{h}$$

$$\binom{\mu}{j} \equiv \frac{\mu(\mu-1) \cdots (\mu-j+1)}{j!} , \quad \binom{\mu}{0} = 1$$

Hermite Interpolation : Prelude

Example:

$f(0) = -1 \qquad f(1) = -1 \qquad f'(1) = +4$

3 conditions $\rightarrow$ quadratic polynomials

write $f(x) = f(0) \cdot M_0(x) + f(1) M_1(x) + f'(1) M_2(x)$

$M_0(x=0) = 1 \qquad M_0(x=1) = 0 \qquad M_0'(x=1) = 0$

$M_1(x=0) = 0 \qquad M_1(x=1) = 1 \qquad M_1'(x=1) = 0$

$M_2(x=0) = 0 \qquad M_2(x=1) = 0 \qquad M_2'(x=1) = 1$

$M_0 = 1 - 2x + x^2$

$M_1 = 2x - x^2$

$M_2 = -x + x^2$

$\therefore f(x) = -1 \cdot (x^2 - 2x + 1) + (-1) \cdot (-x^2 + 2x) + 4(x^2 - x)$

$= 4x^2 - 4x - 1$

· Hermite Interpolation:

An ideal ~~More general~~ situation:

$$p(x_i) = y_i, \quad p'(x_i) = y_i', \quad i = 1, \cdots, n$$

in which $x_1, x_2, \cdots, x_n$ are distinct nodes (real or complex) and $y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n'$ are given data.

$\Rightarrow$ $2n$ conditions imposed $\rightarrow$ look for a polynomial of at most $2n-1$.

$\Rightarrow$ recall Lagrange polynomial

$$l_i(x) \equiv \frac{(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

construct two functions $\cancel{f}$ g & h:

$$g_i(x) = (x-x_i)\left[l_i(x)\right]^2$$

$$h_i(x) = \left[1 - 2l_i'(x_i)(x-x_i)\right]\left(l_i(x)\right)^2$$

For $i = 1, \cdots, n$
$j = 1, \cdots, n$

$$g_i(x_j) = 0 \qquad g_i'(x_j) = \delta_{ij}$$

$$h_i'(x_j) = 0 \qquad h_i(x_j) = \delta_{ij}$$

$$\therefore \; P(x) = \sum_{i=1}^{n} y_i h_i(x) + \sum_{i=1}^{n} y_i' g_i(x)$$

Check :

$$p(x_j) = \sum_{i=1}^{n} y_i h_i(x_j) + \cancel{\sum_{i=1}^{n} y_i' g_i(x_i)}$$

$$= \sum_{i=1}^{n} y_i \delta_{ij} = y_j$$

$$p'(x_j) = \sum_{i=1}^{n} y_i h_i'(x_j) + \sum_{i=1}^{n} y_i' g_i(x_j)$$

$$= 0 + \sum_{i=1}^{n} y_i' \delta_{ij} = y_j'$$

How do we generalize this to the example ?

Let's go back to the example with some generalization:

$f(x_1) = y_1$

$f(x_2) = y_2$

$f'(x_2) = y_2'$

$\rightarrow \; f(x) = y_1 M_1(x) + y_2 M_2(x) + y_2' M_3(x)$

$M_1(x_1) = 1 \qquad M_1(x_2) = 0 \qquad M_1'(x_2) = 0$

$M_2(x_1) = 0 \qquad M_2(x_2) = 1 \qquad M_2'(x_2) = 0$

$M_3(x_1) = 0 \qquad M_3(x_2) = 0 \qquad M_3'(x_2) = 1$

(1st) $M_1(x)$ has double roots $x_2 \Rightarrow M_1(x) = \left(\dfrac{x - x_2}{x_1 - x_2}\right)^2$

2ndly $\quad M_3(x) = (x-x_2)\, l_2(x) = (x-x_2)\dfrac{x-x_1}{x_2-x_1}$ ✓

Finally $\quad M_2(x) = \left[1 - l_2'(x_2)(x-x_2)\right] l_2(x)$

$$= \left[1 - \frac{x-x_2}{x_2-x_1}\right]\frac{x-x_1}{x_2-x_1} \quad ✓$$

Recall the error analysis for polynomial interpolation:

$$f(x) - P_n(x) = \frac{1}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)\, f^{(n+1)}(\xi_n)$$

Define $\Psi_n(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

We can show that $|\Psi_n(x)| \le n!\, h^{n+1}$ for $a \le x \le b$

Solution: suppose $x_j \le x \le x_{j+1}$, $\quad 0 \le j \le n-1$

$|x-x_j| \le h \qquad |x-x_{j+1}| \le h$

$|x-x_{j-1}| \le 2h \qquad |x-x_{j+2}| \le 2h$

$\vdots$

$|x-x_1| \le jh \qquad |x-x_{n-1}| \le |n-1-j|\, h$

$|x-x_0| \le (j+1)h \qquad |x-x_n| \le (n-j)\, h$

$\therefore |\Psi_n(x)| \le (j+1)h \cdot jh \cdots h \cdot h \cdot 2h \cdots (n-1-j)h\,(n-j)h$

$= (j+1)!\,(n-j)!\,h^{n+1}$

The last expression is larger when $j=0$ or $j=n-1$

$\Rightarrow$ either case we have

$$|\Psi_n(x)| \le n!\, h^{n+1} \quad ※$$

Therefore $$|f(x)-P_n(x)| = \frac{1}{(n+1)!}\,|f^{(n+1)}(\xi_n)|\cdot|\Psi_n(x)|$$

$$\boxed{|f(x)-P_n(x)| \le \frac{n!\,h^{n+1}}{(n+1)!}\cdot\left|f^{(n+1)}(\xi_n)\right|}$$

recall that

$$f[x_0, x_1, \ldots, x_n, t] = \frac{1}{(n+1)!}\,f^{(n+1)}(\xi_n)$$

$$\therefore \boxed{|f(x)-P_n(x)| \le n!\,h^{n+1}\,\left|f[x_0, x_1, \ldots, x_n, t]\right|}$$

Special case using Hermite interpolation:

$$p^{(i)}(x_1) = f^{(i)}(x_1), \quad i = 0, 1, \cdots, N-1$$

using $f[\underbrace{x_0, x_0, \cdots x_0}_{n+1}] = \frac{f^{(n)}(x_0)}{n!}$

the Newton divided difference form of the Hermite interpolating polynomial is:

$$p(x) = \cancel{p(x_1)+} \; f(x_1) + (x-x_1) f[x_1, x_1] + (x-x_1)^2 f[x_1, x_1, x_1] + \cdots + (x-x_1)^{n-1} f[\underbrace{x_1, x_1, \cdots x_1}_{n \text{ terms}}]$$

$\Rightarrow$ Basically Taylor series

or the ~~b~~ application of the forward divided difference ~~for~~ to nodes $\{\underbrace{x_1, x_1, \cdots, x_1}_{n \text{ nodes}}\}$

multiple

$\Rightarrow$ generalization of divided difference interpolation

to

(a) general multiple roots

(b) general Hermite polynomial interpolation

(a) say among the distinct $n+1$ nodes

$$x_0, \ x_1, \ \cdots \ x_i, x_{i+1}, \ \cdots \ x_n$$

we have m double roots

$$x_{j1}, \ x_{j2}, \cdots, \ x_{jm}$$

say, without generality, $x_0, \ x_1, \ \cdots \ x_{m-1}$

$\Rightarrow$ generate a new set of nodes:

$$[x_0, x_0, \ x_1, x_1, \ x_2, x_2, \ \cdots \ x_{m-1} \ x_{m-1} \ x_m \ \cdots \ x_n]$$

$n+1+m$ conditions

$n+m$ polynomial

$$\begin{aligned} P_{n+m}(x) = \ & f(x_0) + (x-x_0) f[x_0, x_0] \\ & + (x-x_0)^2 f[x_0, x_0, x_1] \\ & + \cdots \\ & + (x-x_0)^2 (x-x_1)^2 \cdots (x-x_{m-1})^2 f[x_0, \cdots x_m] \\ & + \cdots \\ & + (x-x_0)^2 (x-x_1)^2 \cdots (x-x_{m-1})^2 (x-x_m) \cdots (x-x_n) \\ & \qquad f[x_0, x_0, \cdots x_n, t] \end{aligned}$$

(b) ~~say~~ the same applies to the most general Hermite interpolation:

$x_0, x_1, \cdots x_n$ nodes

among these distinct nodes,

we know the α_j derivatives of the jth node.

$\Rightarrow$ generate a new set of nodes

$$[x_0, \quad , \underbrace{x_j, x_j, x_j}_{\alpha_j}, x_{j+1} \cdots x_n]$$

then use the divided difference formula for interpolation.