M 614 02/16/05

problem 30 P.0

(a) $p(x_0) = f(x_0)$

$p(x_2) = f(x_2)$

$p'(x_1) = f'(x_1)$

$p''(x_1) = f''(x_1)$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_2) \\ f'(x_1) \\ f''(x_1) \end{bmatrix}$$

$A \cdot M = D$

$\det(A) \neq 0$

$\therefore$ M is uniquely found.

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = 2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$= 2 \cdot (-2) = -4 \neq 0$

(b) $E(x) = f(x) - p(x)$

$$G(x) = E(x) - \frac{\Psi(x)}{\Psi(t)} E(t)$$

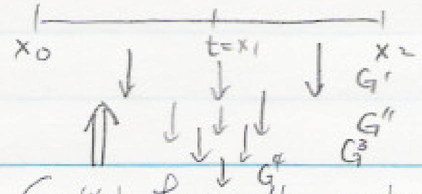
$\Psi(x) = (x^2 - 1)\varphi(x)$

$G'(x_1) = 0 \Rightarrow \varphi(x) = a(x^2 + 1)$

$G''(x_1) = 0$

$\therefore \Psi^{(4)}(x) = a\, 4! \rightarrow a = 1$

$$\therefore G(x) = E(x) - \frac{(x^4 - 1)}{\Psi(t)} E(t)$$




$\boxed{G^{(4)}(\xi_x)=0}$

$\therefore$ $G(x)$ has 4 roots

$\Rightarrow$ everything follows the proof in Theorem 3.2

$G^4(x) = 0$ when $x = \xi_x$

$$E^4(x) = f^4(x)$$

$$G^4(\xi_x) = E^4(\xi_x) - \frac{\Psi^4(\xi_x)}{\Psi(t)} E(t) = 0$$

$$E^4(\xi_x) = \frac{4!}{\Psi(t)} E(t)$$

$$E(t) = \frac{\Psi(t)}{4!} f^4(\xi_x)$$

$$E(t) = \frac{(t^4-1)}{4!} f^4(\xi_x)$$

$$\therefore \quad f(x) - p(x) = \frac{x^4-1}{4!} f^4(\xi_x) \quad \#$$

problem 27

$$E(x) = f(x) - P_n(x)$$

$$G(x) = E(x) - \frac{\Psi(x)}{\Psi(t)} E(t)$$

$\Psi(x) \Rightarrow$ must satisfy

$\Psi^{(j)}(a) = 0 \qquad j = 0, 1, \dots, n-1 \qquad \therefore \Psi(x) = c(x-a)^n(x-b)^n$

$\Psi^{(j)}(b) = 0$

$\Psi^{2n}(x) = \cancel{(n+n)!}\ (2n)! \qquad C = 1$

$$\therefore \Psi(x) = (x-a)^n(x-b)^n$$

$$\therefore \quad G(x) = E(x) - \frac{\Psi(x)}{\Psi(t)} \cdot E(t)$$

$$\therefore \quad G^{2n}(\xi_x) = 0 \Rightarrow E(t) = \frac{(t-a)^n(t-b)^n}{(2n)!} f^{2n}(\xi_x)$$

$$\therefore \quad f(x) - P_n(x) = \frac{(x-a)^n(x-b)^n}{(2n)!} f^{2n}(\xi_x) \quad \#$$

Example: (a) Let $[a,b]$ be a given interval and $a<c<b$.

Define $\sigma_c(x) = \begin{cases} 0 & a \le x \le c \\ (x-c)^3 & c \le x \le b. \end{cases}$

Show that $\sigma_c$ is a spline on $[a,b]$.

(b) Let $x_1 < x_2 \cdots < x_n$ let $p(x)$ be an arbitrary polynomial of degree $\le 3$ and define $S(x) = \sum_{j=2}^{n-1} b_j \sigma_{x_j}(x) + p(x)$

$x_1 \le x \le x_n$

with $b_2, \cdots, b_{n-1}$ arbitrary constants.

Show $S(x)$ is a cubic spline on $[a,b] = [x_1, x_n]$

Solution: (a) $\sigma_c(c) = 0$ , $\sigma_c'(c) = 0$, $\sigma_c''(c) = 0$

$\sigma_c(x)$ is a cubic polynomial for both $a \le x \le c$ & $c \le x \le b$

$\Rightarrow$ $\sigma_c(x)$ is a cubic spline

(b) $p(x)$ is a cubic spline

$\sigma_c(x)$ is a cubic spline from (a)

$\Rightarrow$ linear combination of cubic spline functions must be cubic spline as well.

Note: ① $S'''(x)$ is discontinuous at nodes $x_1, x_2, \ldots, x_n$.

Note ② if $b_j \neq 0$, $S'''(x)$ is discontinuous at $x_j$

$$S'''(x_j^+) - S'''(x_j^-) = 6b_j - 0 = 6b_j$$

B-splines – basic spline function

$$S(x) = P_{m-1}(x) + \sum_{j=1}^{n-1} \beta_j (x - x_j)_+^{m-1}$$

$$(x - x_j)_+^{m-1} = \begin{cases} 0 & x - x_j < 0 \\ (x - x_j)^{m-1} & x \geq x_j \end{cases}$$

$S(x)$ is a spline function of order $m$ with knots $\{x_0, x_1, \cdots, x_n\}$
$P_{m-1}(x)$ is a uniquly chosen polynomial of degree $\leq m-1$
$\beta_1, \beta_2, \ldots, \beta_{n-1}$ are all uniquely determined coefficients.

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An alternate way of writing down the B-splines:

⓪ $f[x_0, x_1, \cdots, x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{\Psi_n'(x_j)}$ $\quad \Psi_n'(x_j) = (x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)$

① $f_x(t) \equiv (t - x)_+^3$ by definition

② $f_x[x_i, x_{i+1}, \cdots, x_{i+4}] = \sum_{j=i}^{i+4} \frac{f_x(x_j)}{\Psi_n'(x_j)} = \sum_{j=i}^{i+4} \frac{(x_j - x)_+^3}{\Psi_n'(x_j)}$

$$\Psi_n(x) = (x - x_i)(x - x_{i+1})(x - x_{i+2})(x - x_{i+3})(x - x_{i+4})$$

$$B_i(x) = (x_{i+4} - x_i) \sum_{j=i}^{i+4} \frac{(x_j - x)_+^3}{\Psi_i'(x_j)}$$

$\Rightarrow$ $B_i(x)$ is a cubic spline with knots $x_i$, $x_{i+1}$, $x_{i+2}$, $x_{i+3}$, $x_{i+4}$

several properties of B-splines are listed in Theorem 3.5 (3.7.36) ~ (3.7.40)

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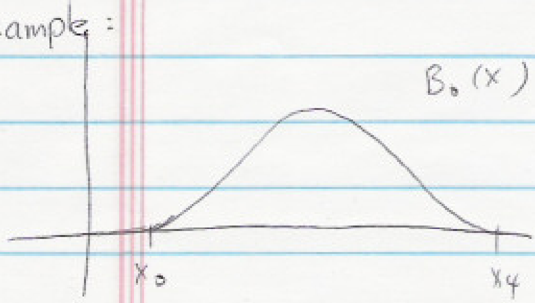
For example:




$B_0(x) = 0$ for $x < x_0$, $x > x_4$

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Introduction to the best approximation theorem & Chebyshev polynomials.

$f$ is a given function, continuous on interval $a \le x \le b$

$p(x)$ is a polynomial

$E(p) \equiv \max_{a \le x \le b} |f(x) - p(x)|$ is the maximum possible error in the approximation of $f(x)$ by $p(x)$

For each degree $n$, define

$$\rho_n(f) = \min_{\deg(p) \le n} E(p)$$

$$= \min_{\deg(p) \le n} \left[ \max_{a \le x \le b} |f(x) - p(x)| \right]$$

$\rho_n(f)$ is the smallest possible value for $E(p)$ that can be attained with a polynomial of degree $\leq n$.

$\Rightarrow$ minmax error.

$\Rightarrow$ A unique polynomial of degree $\leq n$ exists for which the maximum error on $[a, b]$ is $\rho_n(f)$.

$\Rightarrow$ This polynomial is the minimax polynomial approx. of order $n$

$\Rightarrow$ denote it as $m_n(x)$

How to construct $m_n(x)$ ?

Remes algorithm, Least-squares approximation

$\Rightarrow$ Taylor polynomial is not $m_n(x)$.

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A near-minimax approximation method using the Chebyshev polynomials.

Chebyshev polynomials: $T_n(x) = \cos(n \cos^{-1} x)$ $\quad -1 \leq x \leq 1$

(a) $n=0$, $T_0(x) = 1$ $\qquad \theta = \cos^{-1} x$

(b) $n=1$ $\quad T_1(x) = x$

(c) $n=2$ $\quad T_2(x) = \cos(2\theta) = 2\cos^2\theta - 1 = 2x^2 - 1$

$\Rightarrow$ Recursion Relation

$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)$$

$$T_{n+1}(x) = \cos[(n+1)\theta] = \cos n\theta \cdot \cos\theta - \sin n\theta \sin\theta$$

$$+)\ T_{n-1}(x) = \cos[(n-1)\theta] = \cos n\theta \cos\theta + \sin n\theta \sin\theta$$

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos n\theta \cos\theta$$

$$T_{n+1}(x) + T_{n-1}(x) = 2x \cdot T_n(x)$$

Note: ① $|T_n(x)| \le 1$

② $T_n(x) = 2^{n-1}x^n +$ lower-degree terms

modified Chebyshev polynomial:

$$\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x) = x^n + \text{lower-degree terms}$$

$$|\tilde{T}_n(x)| = \frac{1}{2^{n-1}} |T_n(x)| \le \frac{1}{2^{n-1}} \quad \text{for } -1 \le x \le 1,\ n \ge 1$$

Thm: the modified Chebyshev polynomial $\hat{T}_n(x)$ is the degree $n$ monic polynomial with the smallest maximum absolute value on $[-1, 1]$

(a)
We show this ~~ω~~ for $\tilde{T}_3(x)$ by first finding values of $x$ s.t.

$\tilde{T}_3(x) = \pm\frac{1}{4}$ ⟹ from the definition of $T_n(x)$

$$T_3(x) = \cos(3\theta) \qquad \hat{T}_3(x) = \frac{1}{4}\cos 3\theta, \quad \theta = \cos^{-1} x$$

$\therefore \boxed{x_j = \cos\frac{(j-1)\pi}{3}}$ are values of $x$ s.t. $\hat{T}_3(x) = \pm\frac{1}{4}$, $j = 1, 2, 3, 4$

$$\Rightarrow \tilde{T}_3(x_j) = \frac{(-1)^{j-1}}{4} \qquad j = 1, 2, 3, 4$$

(b) assume another monic polynomial $q(x)$

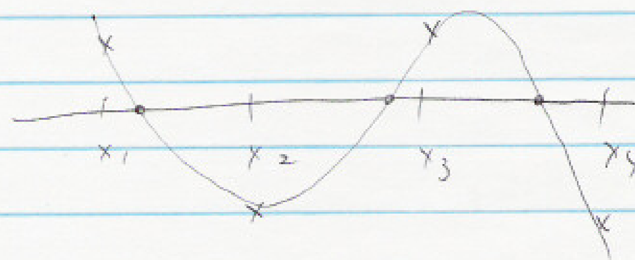
(i) $\max\limits_{-1 \le x \le 1} |q(x)| < \frac{1}{2^{n-1}} = \frac{1}{4}$

(ii) $\deg(q) \le 3$

We define $R(x) \equiv \tilde{T}_3(x) - q(x)$, degree of $R(x) \le 2$

$R(x_1) = \frac{1}{4} - q(x_1) > 0$

$R(x_2) = -\frac{1}{4} - q(x_2) < 0$




$R(x_3) = \frac{1}{4} - q(x_3) > 0$

three roots for $R(x)$ of degree $\le 2$

$R(x_4) = -\frac{1}{4} - q(x_4) < 0$

$\Rightarrow R(x)$ must be zero

$$\therefore \; q(x) = \tilde{T}_3(x)$$

can generalize this to $n \ge 3$.

Therefore $\tilde{T}_n(x)$ is the degree $n$ monic polynomial with the smallest maximum absolute value on $[-1, 1]$. $\left(\frac{1}{2^{n-1}}\right)$

Recall the error in polynomial interpolation

$$f(x) - p_n(x) = \frac{\Psi_n(x)}{(n+1)!} f^{n+1}(\xi_x)$$

$$\Psi_n(x) = (x-x_0)(x-x_1) \cdots (x-x_n)$$

note that $\Psi_n(x)$ is a monic polynomial $x^{n+1}$ has coefficient 1.

$$|f(x) - p_n(x)| = \frac{|\Psi_n(x)|}{(n+1)!} |f^{(n+1)}(\xi_x)|$$

One way to minimize the error on the RHS is to choose grid points so that $|\Psi_n(x)|$ is minimized.

$\Rightarrow$ $\Psi_n(x)$ must be $\tilde{T}_{n+1}(x)$

$\Rightarrow$ this implies the grid points must be roots of $\tilde{T}_{n+1}(x)$:

$$x_j = \cos\left(\frac{2j+1}{2n+2}\pi\right), \quad j = 0, 1, \ldots, n$$

We then obtain the near-minimax polynomial approximation $p_n(x)$ of degree $n$ by interpolating to $f(x)$ at $x_j$ on $[-1, 1]$.

Example: $f(x) = e^x$ on $[-1, 1]$

find the near-minimax approximation of $C_3(x)$

$\Rightarrow$ the nodes must be roots of $T_4(x)$

| | | $f(x_i) = e^{x_i}$ | $f[x_0, \cdots x_i]$ |
|---|---|---|---|
| $x_0 = \cos\left(\frac{\pi}{8}\right)$ | $= 0.923880$ | 2.5190422 | 2.5190442 |
| $x_1 = \cos\left(\frac{3\pi}{8}\right)$ | $= 0.382683$ | 1.4662138 | 1.9453769 |
| $x_2 = \cos\left(\frac{5\pi}{8}\right)$ | $= -0.382683$ | 0.6820288 | 0.7047420 |
| $x_3 = \cos\left(\frac{7\pi}{8}\right)$ | $= -0.923880$ | 0.3969760 | 0.1751757 |

$$C_3(x) = f[x_0] + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3]$$

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Chebyshev polynomial is a set of orthogonal polynomials:

for $n, m \geq 0$,

$$\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2} \delta_{nm}$$ is the orthogonality.

Example: $e^x = \sum_{i=0}^{n} a_i T_i(x)$ $\rightarrow$ chebyshev expansion

$$a_i = \frac{2}{\pi} \int_{-1}^{1} \frac{e^x T_i(x)}{\sqrt{1-x^2}}\, dx$$

HW: compare nea minimax approximation and Chebyshev expansion of $f(x) = e^x$ for $C_1$, $C_3$, & $C_5$ by computing $\max |C_i(x) - e^x|$ on the interval $[-1, 1]$