Approximation of a function f : C^{n+1} on $[a, b]$

- Taylor series : $t_n(x)$

$$f(x) - t_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{n+1}(\xi_x)$$

- polynomial interpolation : $p_n(x)$

$$f(x) - p_n(x) = \frac{\Psi_n(x)}{(n+1)!} f^{n+1}(\xi_x)$$

- near minimax approximation :

  use the Chebyshev zeros for nodes

  minimizing $|\Psi_n(x)|$ or the Chebyshev zeros

- least square approximation :

  Chebyshev expansion

$$C_n(x) = {\sum_{j=0}^{n}}' C_j T_j(x) \qquad C_j = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\, T_j(x)}{\sqrt{1-x^2}}\, dx$$

first term should be halved in the summation

- minimax approximation $q_n^*(x)$

$$f(x_j) - q_n^*(x_j) = \sigma(-1)^j \rho_n(f) \quad , \text{ where } \rho_n(f) \equiv \| f - q^* \|_\infty$$

$j = 0, 1, \cdots, n+1$ &

$a \le x_j \le b$, $\sigma = \pm 1$ depending only on f and n

NB1 In many applications, least-square approximation is used.

NB2 In particular, Chebyshev expansion is used when Gibbs phenomenon is found on equispaced grid points.

NB3
From the Chebyshev equioscillation theorem, we would expect $C_n(x)$ to be close to the minimax approximation $q_n^*(x)$.

From previous lecture, we know that an approximation will be "close" to minimax approx. if ① the error oscillates between nodes
② error also evenly distributes among nodes

Based on the above observation, we want to find nodes so that the error oscillates between nodes in a uniform fashion.

Let $f(x) \in C[-1, 1]$ and $F_n(x) = \sum_{k=0}^{n}{}' C_{n,k} T_k(x) \quad -1 \le x \le 1$

Let $-1 \le x_{n+1} < x_n < \cdots < x_1 < x_0 \le 1$

We want to determine coefficients C_{nk} so that the error

$f(x) - F_n(x)$ oscilates in the manner described above.

$$f(x_i) - F_n(x_i) = (-1)^i E_n \, . \qquad i = 0, 1, \cdots, n+1$$

So altogether we've got $n+1+1$ unknowns

$$\underset{C_j,}{\downarrow} \quad \underset{E_n}{\downarrow}$$

To pick the nodes, we observe that $f(x) - F_n(x) \sim C_{n+1} T_{n+1}(x)$

~~if~~ assuming $C_{n+1} T_{n+1}(x)$ is nearly the minimax error.

$T_{n+1}(x_j) = \pm 1$ would be the maximum & minimum of T_{n+1} at nodes $x_j = \cos\left(\frac{j\pi}{n+1}\right) \quad j = 0, 1, \cdots, n+1.$

Note that $T_k(x_i) = \cos(k \cdot \cos^{-1}(x_i)) = \cos\left(\frac{ki\pi}{n+1}\right)$

$$\therefore f(x_i) = \sum_{k=0}^{n}{}' C_{n,k} \cos\left(\frac{ki\pi}{n+1}\right) + (-1)^i E_n$$

set $C_{n,n+1}/2 \equiv E_n$.

$$f(x_i) = \sum_{k=0}^{n}{}'' \; C_{n,k} \cos\left(\frac{ki\pi}{n+1}\right) \qquad \text{———— } (*)$$

because $\cos\left(\frac{i \cdot (n+1)\pi}{n+1}\right) = (-1)^i$

Multiplying $(*)$ by $\cos\left(\frac{ij\pi}{n+1}\right)$ for some $0 \leq j \leq n+1$ and summing over $i \Rightarrow$

$$\sum_{k=0}^{n+1}{}'' \; C_{n,k} \underbrace{\sum_{i=0}^{n+1}{}'' \cos\left(\frac{ij\pi}{n+1}\right)\cos\left(\frac{ik\pi}{n+1}\right)} = \sum_{i=0}^{n+1}{}'' f(x_i)\cos\left(\frac{ij\pi}{n+1}\right)$$

$$\begin{cases} n+1 & \text{if } \begin{cases} j=k=0, \\ j=k=n+1 \end{cases} \\ \frac{n+1}{2} & \text{if } 0<j=k<n+1 \\ 0 & \text{if } j \neq k \\ & \quad 0 \leq j,k \leq n+1 \end{cases}$$

$$\therefore \quad C_{n,j} \cdot \frac{n+1}{2} = \sum_{i=0}^{n+1}{}'' f(x_i) \cos\left(\frac{ij\pi}{n+1}\right)$$

$$\boxed{C_{n,j} = \frac{2}{n+1} \sum_{i=0}^{n+1}{}'' f(x_i) \cos\left(\frac{ij\pi}{n+1}\right)}$$

Thus $\boxed{E_n = \frac{1}{n+1} \sum_{i=0}^{n+1}{}'' (-1)^i f(x_i)}$

How is $C_{n,j}$ related to C_j in the Chebyshev expansion?

Recall that $C_j \equiv \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos(j\theta)\,d\theta$

$$\doteq \frac{2}{\pi}\sum_{i=0}^{n+1}{}'' f\left(\cos\left(\frac{i\pi}{n+1}\right)\right)\cdot\cos\left(\frac{ji\pi}{n+1}\right)\cdot\frac{\pi}{n+1} = C_{n,j}$$

this $\doteq$ is from the trapezoidal rule that we will talk about in Chapter 5.

$\Rightarrow$ If the Chebyshev coefficient $|C_n| \to 0$ as $n \to \infty$ fast enough, then the approximation $F_n(x) \to C_n(x)$ and is a lot easier to calculate.

Example: Let $q(x)$ be a polynomial of degree $\leq n-1$, and

consider $\max\limits_{-1 \leq x \leq 1} | x^n - q(x) |$

What is the smallest possible value for this quantity?

Solve for the $q(x)$ for which the smallest value is attained.

Observe $x^n - q(x)$ is a monomial polynomial of order n

$\Rightarrow \min \max\limits_{-1 \leq x \leq 1} | x^n - q(x) |$ is achieved if

$$x^n - q(x) = + \tilde{T}_n(x)$$

$$\therefore \; x^n - \tilde{T}_n(x) = q(x)$$

Use this example to find the minimax approximation

to $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

using a polynomial of order $n-1$.

$$f^*(x) = \frac{f(x)}{a_n} = x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \cdots + \frac{a_0}{a_n}$$

$q^*_{n-1}(x)$ is the minimax approx.

$\min \max\limits_{-1 \le x \le 1} |f^*(x) - q^*_{n-1}(x)|$ is attained

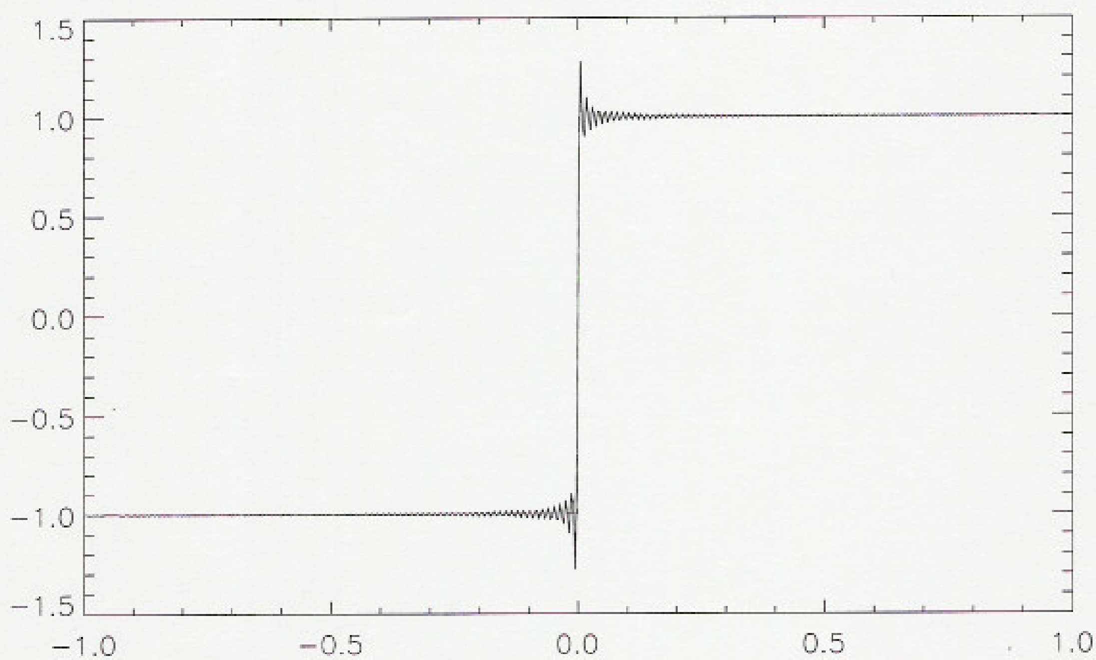
if $\quad f^*(x) - q^*_{n-1}(x) = \hat{T}_n(x)$

$$f^*(x) - \hat{T}_n(x) = q^*_{n-1}(x)$$

$$\therefore \quad f(x) - a_n \hat{T}_n(x) = a_n q^*_{n-1}(x)$$

$\therefore$ the nodes $-1 = x_0 < x_1 < \cdots < x_n = 1$
must be the Chebyshev extreme points for $T_n(x)$

and the error $\rho(f(x)) = \frac{a_n}{2^{n-1}}$ $\#$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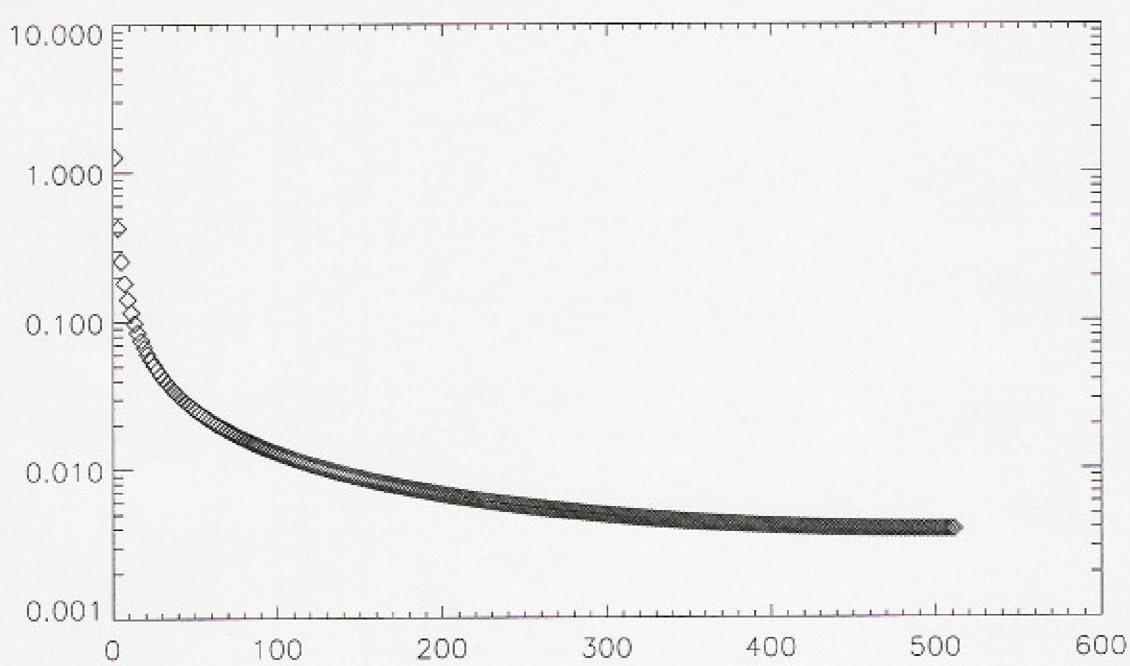
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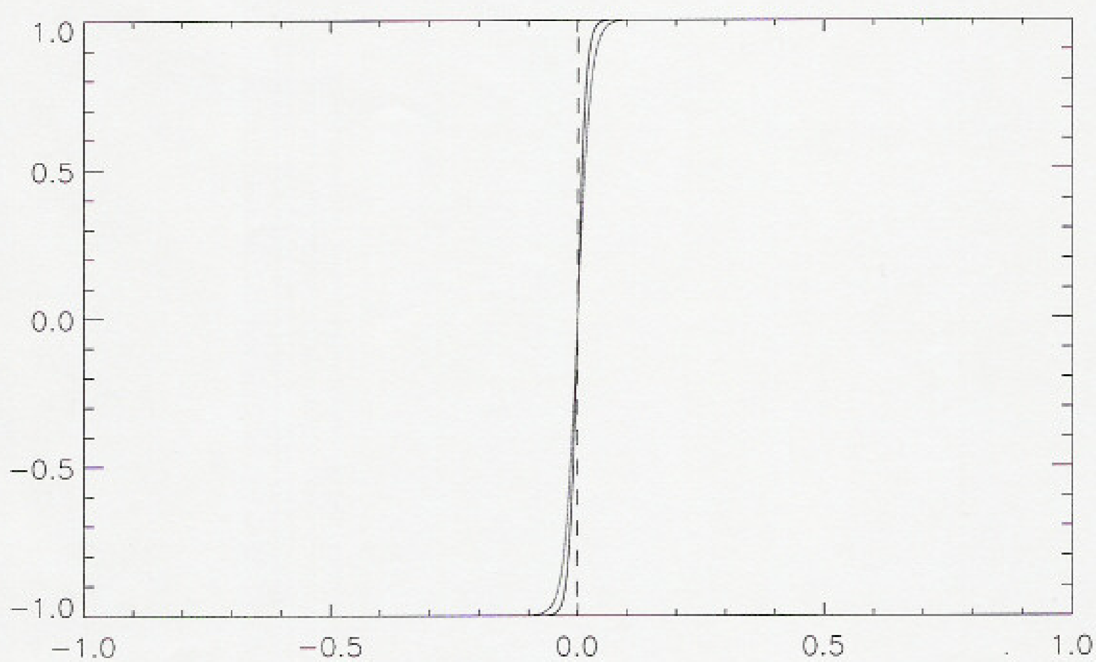
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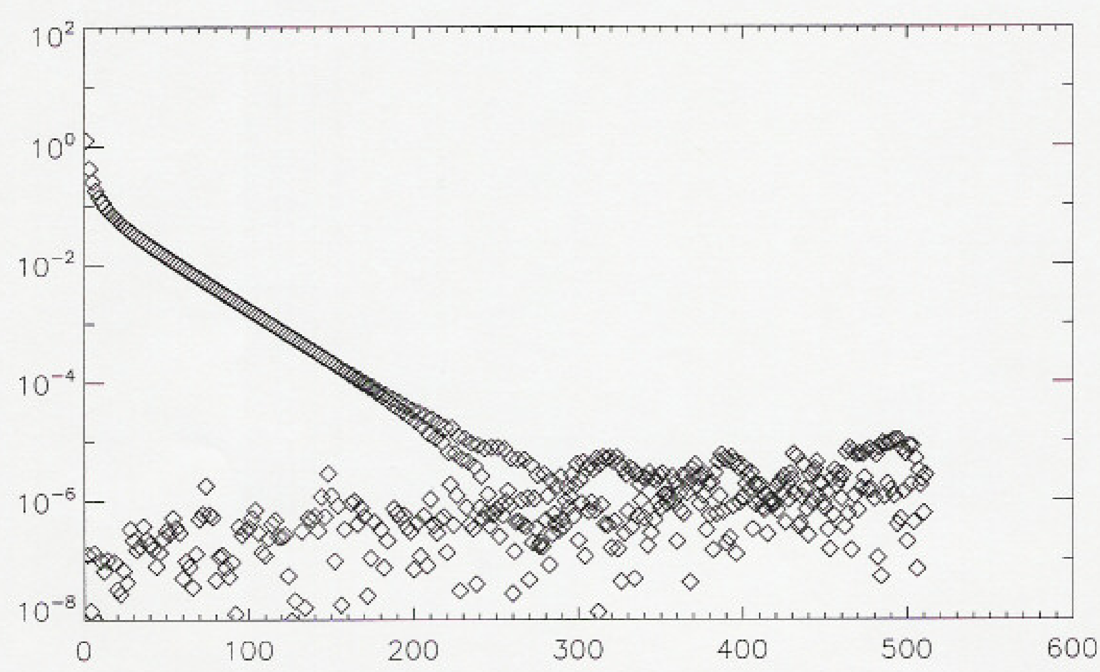
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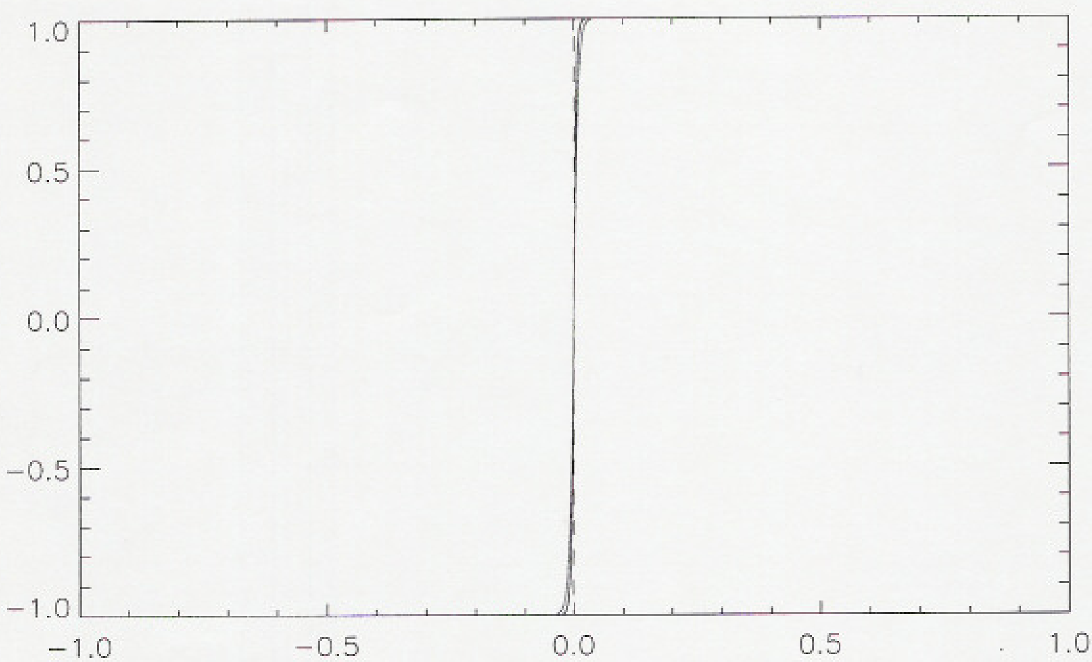
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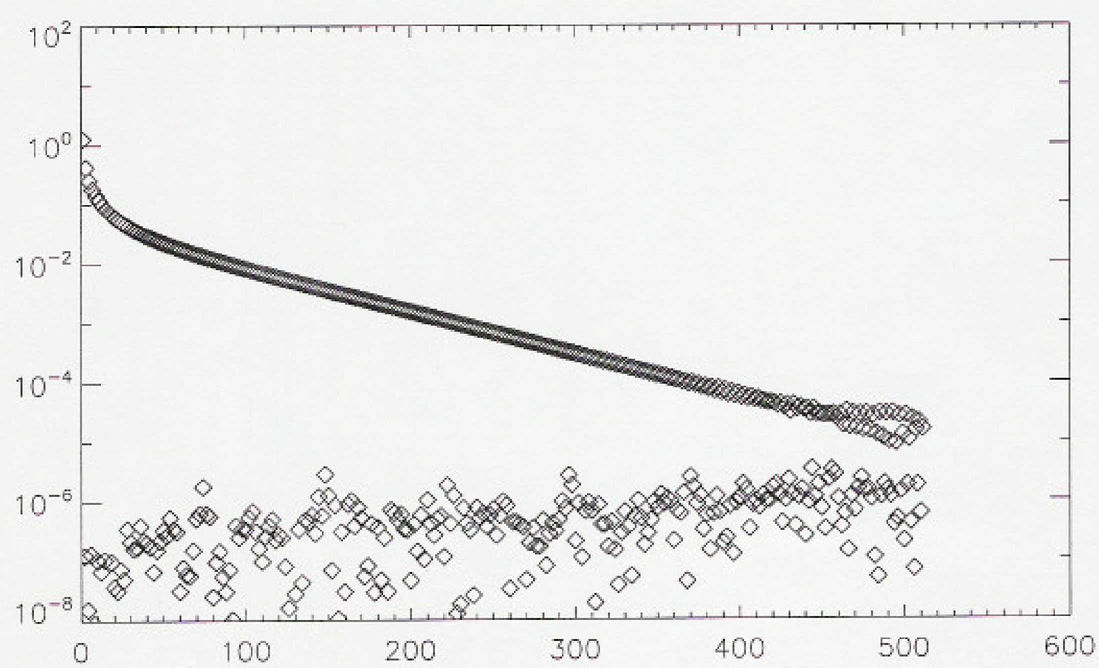
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