Elementary Siphons of Petri Nets and Deadlock Control

Zhiwu Li[†] and Mengchu Zhou[‡], *Fellow, IEEE*

[†]School of Electro-Mechanical Engineering
Xidian University
Xi'an, 710071 China
Email: zhwli@xidian.edu.cn

[‡]Department of Electrical and Computer Engineering
New Jersey Institute of Technology
Newark, NJ 07102 USA
Email: zhou@njit.edu

Abstract–The importance of siphons is well recognized in the detection and analysis of deadlocks in a Petri net. Based on them, a variety of techniques have been developed for deadlock problems in discrete event systems. Since the number of siphons in a net is theoretically exponential with its size, the major disadvantage of the existing siphon-based approaches is that the number of siphons that have to be considered is large or grows fast as these methods proceed, which inevitably leads to structurally complex deadlock-free Petri net supervisors. To minimize the number of siphons that have to be controlled, this paper divides siphons into elementary and redundant ones. It is shown that the number of the former is bounded by the smaller of place count and transition count in a net. We formulate the conditions under which a redundant siphon can be always marked if its elementary siphons are controlled. An algorithm is developed to find the set of elementary siphons in a net system. Some interesting and open problems concerning elementary siphons are discussed in detail. This work is of significance to reduce the structural complexity of the analysis and design of liveness enforcing Petri net supervisors for discrete event systems.

Index Terms–Petri net, siphon, elementary siphon, deadlock prevention, discrete event systems.

1 Introduction

Petri nets [21] [27] are a mathematical tool well suited for modelling and analyzing discrete event systems exhibiting behavior such as concurrency, conflict, and causal dependency between events. Among many properties of Petri nets, liveness that implies deadlock freedom is of outstanding significance. The analysis of the properties can be performed via reachability analysis, structural analysis, reduction, and invariant-analysis at the decreasing computational requirements and analysis power. In the second method, the siphon-related theory has been well studied and is mature for many classes of Petri nets such as free choice Petri nets [6], augmented marked graphs [5], and ERCN-merg nets [26]. A siphon is a subset of places that once unmarked will never receive new tokens. It is said to be controlled if for each reachable marking it remains sufficiently marked. When all siphons are controlled, the net is deadlock-free. Deadlock analysis and control techniques based on the structural theory of Petri nets aim at finding a relationship between liveness of a net and its structure. Many deadlock prevention policies characterize deadlock behavior of the system in terms of siphons and utilize this characterization to control the system [23] [4] [7] [11].

Deadlock prevention is usually achieved by using an off-line mechanism to control the requests for resources to ensure that deadlocks never occur. Monitors and related arcs are added to the original net to achieve such purposes [1] [2] [3] [7] [12] [18] [19] [25] [10] [11] [22]. Each possibly emptiable minimal siphon requires a monitor to be added to prevent itself from being emptied. Unfortunately the number of such siphons in a net grows exponentially with its size [8] [17]. Hence, the shortcoming of the existing methods is that they need introduce too many monitors when the number of such siphons is large, leading to a much more complex Petri net supervisor than the original plant net model. To overcome the shortcoming, this paper presents several novel siphon concepts and related theory so that the number of monitors and arcs can be drastically reduced while achieving the same or better control results.

We present in the next section the basic Petri net definitions and notation used throughout this paper. Some new concepts such as nominal, equivalent, redundant, and elementary siphons are introduced in Section 3. Section 4 focuses on the investigation of the controllability of redundant siphons, which mainly shows the conditions under which a redundant siphon is controlled via its elementary siphons. Section 5 presents an algorithm to find the set of elementary siphons in a net system for deadlock control purpose. The concepts and techniques proposed in this paper are illustrated by an example in Section 6. In Section 7, the application of elementary siphons to deadlock control is given via a flexible manufacturing system. Section 8 concludes the paper by discussing some interesting and open problems owing to the discovery of elementary siphons in Petri nets.

2 Preliminaries of Petri Nets

A Petri net [21] [27] is a 4-tuple $N = (P, T, F, W)$ where P and T are finite, nonempty, and disjoint sets. P is the set of places and T is the set of transitions. $F \subseteq (P \times T) \cup (T \times P)$ is called flow relation or the set of directed arcs. $W : F \to \mathbf{N}$ is a mapping that assigns a weight to an arc, where $\mathbf{N} = \{1, 2, \ldots\}$. $N = (P, T, F, W)$ is called an ordinary Petri net, denoted as $N = (P, T, F)$ if $\forall f \in F, W(f) = 1$. Unless otherwise stated, we consider only ordinary Petri nets in this paper. The preset of a node $x \in P \cup T$ is defined as $^\bullet x = \{y \in P \cup T \mid (y, x) \in F\}$. The postset of a node $x \in P \cup T$ is defined as $x^\bullet = \{y \in P \cup T \mid (x, y) \in F\}$. The preset (postset) of a set is defined as the union of the presets (postsets) of its elements.

A marking of $N = (P, T, F)$ is a mapping $M : P \to \mathbf{IN}$, where $\mathbf{IN}$=$\{0, 1, 2, \ldots\}$. (N, M) is called a net system or a marked net. A transition $t \in T$ is enabled under M, in symbols $M[t\rangle$, if and only if $\forall p \in^\bullet t : M(p) > 0$ holds. If $M[t\rangle$ holds the transition t may fire, resulting in a new marking M', denoted by $M[t\rangle M'$, with $M'(p) = M(p) - 1$ if $p \in^\bullet t \backslash t^\bullet$; $M'(p) = M(p) + 1$ if $p \in t^\bullet \backslash^\bullet t$; and otherwise $M'(p) = M(p)$, for all $p \in P$. For net N, the set of all markings reachable from a marking M_0, in symbols $R(N, M_0)$, is the smallest set in which $M_0 \in R(N, M_0)$ and $M' \in R(N, M_0)$ if both $M \in R(N, M_0)$ and $\exists t : M[t\rangle M'$ hold. A transition $t \in T$ is live under M_0 if and only if $\forall M \in R(N, M_0)$, $\exists M' \in R(N, M) : M'[t\rangle$ holds. N is dead under M_0 if and only if $\nexists t \in T : M_0[t\rangle$ holds. (N, M_0) is deadlock-free if and only if $\forall M \in R(N, M_0)$, $\exists t \in T : M[t\rangle$ holds. (N, M_0) is live if and only if $\forall t \in T : t$ is live under M_0. A transition sequence $\sigma = t_1 t_2 \ldots t_n$ is said to be firable at marking M if there exist markings $M_1, M_2, \ldots, M_n$ such that $M[t_1\rangle M_1[t_2\rangle M_2 \ldots M_{n-1}[t_n\rangle M_n$ holds. (N, M_0) is bounded if and only if $\exists k \in \mathbf{N}$, $\forall M \in R(N, M_0)$, $\forall p \in P : M(p) \leq k$ holds. $N = (P, T, F)$ is pure if and only if $\nexists (x, y) \in (P \times T) \cup (T \times P) : (x, y) \in F \wedge (y, x) \in F$. We assume that in the following all Petri nets are bounded and pure.

A P-vector is a column vector $I : P \to Z$ indexed by P and a T-vector is a column vector $J : T \to Z$ indexed by T, where Z is the set of integers. The incidence matrix of N is a matrix $[N] : P \times T \to Z$ indexed by P and T such that $[N](p, t)$=-1 if $p \in^\bullet t \backslash t^\bullet$; $[N](p, t)$=1 if $p \in t^\bullet \backslash^\bullet t$; and otherwise $[N](p, t)$=0 for all $p \in P$ and $t \in T$. We denote column vectors where every entry equals 0(1) by $\mathbf{0}(\mathbf{1})$. I^T and $[N]^T$ are the transposed versions of a vector I and a matrix $[N]$, respectively.

I is a P-invariant (place invariant) if and only if $I \neq 0$ and $I^T \cdot [N]$=$\mathbf{0}^T$ hold. P-invariant I is called a P-semiflow if its every entry is greater than or equal to zero. $\parallel I \parallel = \{p \in P \mid I(p) \neq 0\}$ is called the support of I. I is called a minimal P-invariant if

$\| I \|$ is not a superset of the support of any P-invariant.

A nonempty set $S \subseteq P$ is a siphon if and only if ${}^\bullet S \subseteq S^\bullet$ holds. A siphon is minimal if and only if there is no siphon contained in S as a proper subset. Unless otherwise stated, a siphon refers to a minimal one in the following discussions. Siphon S is said to be a strict minimal one if and only if S is minimal, ${}^\bullet S \subset S^\bullet$ and ${}^\bullet \neq S^\bullet$. If siphon S is the support of a P-invariant and is initially marked, then it can never be emptied. $M(p)$ indicates the number of tokens on p under M. p is marked by M if and only if $M(p) > 0$. A subset $D \subseteq P$ is marked by M if and only if at least one place in D is marked by M. The sum of tokens on all places in D is denoted by $M(D)$, where $M(D) = \sum_{p \in D} M(p)$. An empty siphon with respect to a Petri net marking M is a siphon S such that $\sum_{p \in S} M(p) = 0$.

Let (N, M_0) be a net system with $N = (P, T, F)$, I be a P-invariant, and $S \subseteq P$ be a siphon of N. S is said to be potentially invariant-controlled by P-invariant I if $\{p \mid I(p) > 0\} \subseteq S$. Siphon S is controlled by P-invariant I under M_0 if and only if $I^T \cdot M_0 > 0$ and $I(p) \leq 0$ for all $p \in P \backslash S$ hold, or equivalently, $I^T \cdot M_0 > \mathbf{0}$ and $\{p \in P \mid I(p) > 0\} \subseteq S$ [18]. Such a siphon is also called an invariant-controlled siphon. S is said to be uncontrollable if $M_0(S) = 0$. S is said to be controllable if $M_0(S) > 0$. And S is said to be controlled if $\forall M \in R(N, M_0)$, $M(S) > 0$.

The following facts are known [5-7, 17-18, 21, 27]. (1) If I is a P-invariant of N then $\forall M \in R(N, M_0) : I^T \cdot M = I^T \cdot M_0$. (2) Let $S \subseteq P$ be a siphon of N. If S is controlled by a P-invariant I under M_0, S cannot be emptied, i.e., $\forall M \in R(N, M_0), S$ is marked under M. (3) A siphon S free of tokens at a marking remains token-free. Furthermore, all transitions connected to S are not live. (4) For any marking such that no transition is enabled, the set of empty places forms a siphon. (5) A Petri net is deadlock-free if no (minimal) siphon eventually becomes empty.

3 Elementary Siphons in Petri Nets

3.1 Nominal Siphons

Definition 1 Let (N, M_0) be a net system and S be a siphon of N. S is called a nominal siphon if $\forall M \in R(N, M_0), M(S) > 0$.

Obviously, a nominal siphon can always be marked. To avoid explicit state enumeration, an effective way to find nominal siphons in a net system is an integer programming technique. Let (N, M_0) be a net system and $I_1, I_2, \ldots,$ and I_m ($m \in \mathbf{N}$) be its minimal P-semiflows. By solving the following integer programming problem,

$$min\{M(S)\}$$
$$\text{subject to}$$
$$\begin{cases} I_1^T \cdot M = I_1^T \cdot M_0 \\ I_2^T \cdot M = I_2^T \cdot M_0 \\ \ldots\ldots \\ I_m^T \cdot M = I_m^T \cdot M_0 \end{cases}$$

we can see that S is a nominal siphon if $min\{M(S)\} \geq 1$. This is a sufficient condition for the non-emptiness of S since the set of markings derived from P-semiflows is generally a superset of the set of reachable markings in a Petri net system. It is known that the general integer programming problems are NP-complete. The determination of a nominal siphon can be transformed into the problem of solving a set of inequalities. If the following set of inequalities has no solution, S is hence a nominal siphon.

$$\begin{cases} \sum_{p\in S} M(p) \leq 0 \\ I_1^T \cdot M = I_1^T \cdot M_0 \\ I_2^T \cdot M = I_2^T \cdot M_0 \\ \ldots\ldots \\ I_m^T \cdot M = I_m^T \cdot M_0 \end{cases}$$

By Lenstra's algorithm [24], for each fixed natural number n, there exists a polynomial algorithm which can find an integral solution for a given rational system $Ax \leq b$, in n variables or decides that no such solution exists, where A is a matrix and b is a column vector. This means that a polynomial algorithm can be developed to decide whether or not a siphon is a nominal one.

3.2 Equivalent Siphons

In this subsection, we define equivalent siphons. Different numbers of tokens initially marked at equivalent siphons lead to the concepts of poor and rich siphons.

Definition 2 Let $S \subseteq P$ be a subset of places of Petri net $N = (P, T, F)$. P-vector λ_S is called the characteristic P-vector of S if and only if $\forall p \in S : \lambda_S(p)$=1; otherwise $\lambda_S(p) = 0$.

Definition 3 Let $S \subseteq P$ be a subset of places of N. η_S is called the characteristic T-vector of S if $\eta_S^T = \lambda_S^T \bullet [N]$.

Definition 4 Let S_1 and S_2 be two siphons in a net. S_1 and S_2 are said to be equivalent, denoted by $S_1 \cong S_2$, if $\eta_{S_1} = \eta_{S_2}$ holds.

Definition 5 Let Π be the set of siphons in a net N. $\langle S\rangle \subseteq \Pi$ is called a set of equivalent siphons if (1) $\forall S', S'' \in \langle S\rangle, \eta_{S'} = \eta_{S''}$ and (2) $\forall S' \in \langle S\rangle, \forall S'' \in \Pi\backslash\langle S\rangle, \eta_{S'} \neq \eta_{S''}$.

Proposition 1 Let $\langle S\rangle$ be a set of equivalent siphons in a net. We define relationship $R = \{(S', S'') \mid S', S'' \in \langle S\rangle\}$. Thus R is an equivalent relationship on $\langle S\rangle$.

Proof: We have to prove that R is reflexive, symmetric, and transitive. It is easy to see that $\forall S^\nabla, S'$, and $S'' \in \langle S\rangle$, one can get (a) $\eta_{S^\nabla} = \eta_{S^\nabla}$, (b) $\eta_{S^\nabla} = \eta_{S'} \Rightarrow \eta_{S'} = \eta_{S^\nabla}$, and (c) $\eta_{S^\nabla} = \eta_{S'} \wedge \eta_{S'} = \eta_{S''} \Rightarrow \eta_{S^\nabla} = \eta_{S''}$. This proposition trivially holds. □

It is easy to see that $\langle S\rangle$ is an equivalent class of Π.

Corollary 1 Let S and $S' \in \Pi$ be two siphons of a net. We have (1)$\langle S\rangle \neq \phi$ and $\langle S\rangle \subseteq \Pi$; (2)$\langle S\rangle = \langle S'\rangle$ if $(S, S') \in R$; (3)$\langle S\rangle \cap \langle S'\rangle = \phi$ if $(S, S') \notin R$; (4) $\cup_{S\in\Pi} \langle S\rangle = \Pi$; (5) Let π be the set of equivalent classes of the elements in Π. π is a partition of Π.

Definition 6 S is said to be a poor siphon in (N, M_0) if $\nexists S' \in \langle S\rangle$ such that $M_0(S') < M_0(S)$ holds.

Obviously, if $\langle S\rangle$ has only one siphon, the siphon is poor.

Corollary 2 If siphons S and S' are poor, $M_0(S) = M_0(S')$.

Definition 7 $S' \in \langle S\rangle$ is called a rich siphon if it is not poor in $\langle S\rangle$.

Corollary 3 A rich siphon is nominal.

Proof: Let Π be the set of siphons in N. If S is rich, there must exist a siphon $S' \in \Pi$ such that both $\eta_{S'} = \eta_S$ and $M_0(S) > M_0(S')$ hold. Hence the firing of any transition removes the same number of tokens from S and S'. For marking $M \in R(N, M_0)$ where $M(S') = 0$, we have $M(S) > 0$ due to $M_0(S) > M_0(S')$. At marking M, no transition which can remove tokens from S is enabled again since S' is emptied. Hence the least number of tokens staying at S is $M_0(S) - M_0(S') > 0$. This leads to the truth of this corollary. □

Corollary 4 Let S' and S'' be two poor siphons in $\langle S \rangle$. S' is controlled iff S'' is controlled.

Proof: Assume that S' and S'' be two poor siphons in $(N, M_0), N = (P, T, F)$. Therefore, they have the identical characteristic T-vector and the same number of initial tokens. $\forall t \in T$, the number of tokens removed from S' is identical with that from S'' if t fires. The controllability of S' means the least number of tokens in S' is greater than 0. Hence, the least number of tokens staying in S'' is greater than 0. S'' is therefore controlled. □

3.3 Elementary and Redundant Siphons

In this subsection, we present the definitions of elementary and redundant siphons in a Petri net. Note that the terminology of elementary and redundant siphons first appeared in the authors' previous work [19]. However, in this paper, we re-define them in a more understandable way.

Definition 8 Let $N = (P, T, F)$ be a net with $\mid P \mid= m, \mid T \mid= n$ and k siphons, $S_1, S_2, \ldots,$ and $S_k, m, n, k \in \mathbf{N}$. Let $\lambda_{S_i}(\eta_{S_i})$ be the characteristic $P(T)$-vector of siphon $S_i, i \in \{1, 2, \ldots, k\}$. We define $[\lambda]_{k\times m} = [\lambda_{S_1} \mid \lambda_{S_2} \mid \ldots \mid \lambda_{S_k}]^T$ and $[\eta]_{k\times n} = [\lambda]_{k\times m} \times [N]_{m\times n} = [\eta_{S_1} \mid \eta_{S_2} \mid \ldots \eta_{S_k}]^T$. $[\lambda]([\eta])$ is called the characteristic $P(T)$-vector matrix of the siphons in N.

Definition 9 Let η_{S_α}, η_{S_β}, ..., and η_{S_γ} $(\{\alpha, \beta, \ldots, \gamma\} \subseteq \{1, 2, \ldots, k\})$ be a linearly independent maximal set of matrix $[\eta]$. Then $\{S_\alpha,\ S_\beta,\ \ldots,\ S_\gamma\}$ is called a set of elementary siphons in N.

Sometimes, S_α, S_β, ..., S_γ are called a base of siphons of net N. By Definition 9, the sets of elementary siphons are not generally unique in a net. Considering the initial marking of a net system, we will develop an algorithm to find the set of elementary siphons for deadlock control purpose.

Definition 10 Let $\{S_\alpha, S_\beta, \ldots, S_\gamma\}$ be the set of elementary siphons and $S \notin \{S_\alpha,\ S_\beta,\ \ldots,\ S_\gamma\}$ be a siphon in net N. Then S is called a strict redundant siphon if $\eta_S = \sum_{i\in\{\alpha,\beta,\ldots,\gamma\}} a_i \cdot \eta_{S_i}$ holds, where $a_i \geq 0$.

Definition 11 Let $\{S_\alpha, S_\beta, \ldots, S_\gamma\}$ be the set of elementary siphons and S be a siphon in net N. S is called a slack redundant siphon if $\{S_\alpha,\ S_\beta,\ \ldots,\ S_\gamma\}$=$\{S_1,\ S_2,\ \ldots,\ S_u\} \cup \{S_{u+1},\ \ldots,\ S_v\}$, $\{S_1,\ S_2,\ \ldots,\ S_u\} \cap \{S_{u+1},\ \ldots,\ S_v\} = \emptyset$, and η_S=$\sum_{i=1}^{u} a_i \cdot \eta_{S_i}$-$\sum_{j=u+1}^{v} a_j \cdot \eta_{S_j}$ hold, where $\forall i \in \{1, 2, \ldots, u\}$, $a_i \geq 0$; $\forall j \in \{u+1, \ldots, v\}, a_j > 0$.

Corollary 5 Let N^S_{ES} be the number of sets of elementary siphons in N. Then we have $N^S_{ES} \leq C_k^{R([\eta])}$, where $R([\eta])$ is the rank of $[\eta]$ and $C_k^{R([\eta])} = k!/((k - R([\eta]))! \cdot R([\eta])!)$.

Proof: This corollary follows due to Definition 9 and the definition of the rank of a matrix. □

Theorem 1 The number of elements in any set of elementary siphons in net N equals the rank of $[\eta]$.

Proof: Assume that N has k siphons and k' elementary siphons. Clearly, we have $k \geq k'$. When there is no redundant siphon in a net, we have $k = k'$. Otherwise, there are $k - k'$ redundant siphons in N. By Definitions 9-11, $\eta_{S_i}(i = k'+1, k'+2, \ldots, k)$ can be linearly represented by $\eta_{S_j}(j = 1, 2, \ldots, k')$. According to the definition of the rank of a matrix, we have that the rank of $[\eta]_{k\times n}$ is k'. □

Theorem 2 Let N_{ES} be the number of elementary siphons in net $N = (P, T, F)$. Then we have $N_{ES} \leq min\{\mid P \mid, \mid T \mid\}$.

Proof: Let k be the number of siphons in N. Since $[\lambda]_{k\times|P|} \times [N]_{|P|\times|T|}$=$[\eta]_{k\times|T|}$, one can get $R([\lambda]_{k\times|P|})$+$R([N]_{|P|\times|T|})$$- \mid P \mid \leq R([\eta]_{k\times|T|}) \leq min\{R([\lambda]_{k\times|P|}), R([N]_{|P|\times|T|})\} \leq R([N]_{|P|\times|T|})$. Thus, $N_{ES} = R([\eta]_{k\times|T|}) \leq R([N]_{|P|\times|T|}) \leq min\{\mid P \mid, \mid T \mid\}$, which leads to the truth of this theorem. □

This result indicates that the number of elementary siphons in a Petri net is bounded by the cardinality of the transition set and place set.

Corollary 6 If $[\lambda]$ is a full-column rank matrix in net N. Then $N_{ES} = R([N])$.

Proof: If $[\lambda]$ is of full-column rank, $R([\lambda]) =\mid P \mid$ holds. Thus, one can get $R([N]) \leq N_{ES} \leq min\{\mid P \mid, R([N])\}$. Due to $R([N]) = min\{\mid P \mid, \mid T \mid\}$, we can have $N_{ES} = R([N])$.

To further investigate the relationship between the number of siphons in a Petri net and that of elementary ones, we randomly generate some Petri nets, in which the number of transitions is fixed at 20 and the incidence relationships are random. The results are shown in Table 1, from which we can see that as $|P|$ increases, the number of elementary siphons fluctuates, in uptrend and slightly but is bounded by $|P|$ and $|T|$. On the other hand, the number of siphons grows rather fast to an unmanageable size.

Table 1: The number of siphons in randomly-generated Petri nets.

Petri Net			No. of siphons	
|P|	|T|	|F|	minimal siphons	elementary siphons
5	20	41	1	1
10	20	46	1	1
20	20	58	17	7
30	20	67	129	15
40	20	88	475	17
50	20	103	1204	15
60	20	122	3221	17
70	20	142	60922	18
80	20	162	92695	19

Let S be a (strict or slack) redundant siphon. In the sequel, if η_S can be linearly represented by elementary siphons' characteristic vectors η_{S_1}, η_{S_2}, ..., and η_{S_n} with non-zero coefficients, we may say that S_1, S_2, ..., S_n are the elementary siphons of S. The sets of elementary and redundant ones are denoted by Π_E and Π_R, respectively. Obviously, we have $\Pi = \Pi_E \cup \Pi_R$.

4 Controllability of Redundant Siphons

A siphon in a net system can be controlled by a P-invariant or a marked trap in it. For an uncontrolled but controllable siphon, we can use an active control method to make it invariant-controlled by adding a monitor. This section mainly focuses on the development of general conditions under which a redundant siphon can always be marked when its elementary siphons are invariant-controlled. First, we present such an active control method for a controllable but uncontrolled siphon. As a convention, in $I = (0, \ldots, 0, 1_{p_x}, \ldots, 1_{p_y}, 0, \ldots, 0, -b_{p_\alpha}, \ldots, -b_{p_\beta}, 0, \ldots, 0, -1_{V_S})^T$, 1_{p_x} means number 1 at the position of p_x and b_{p_α} a positive integer variable in the position of p_α.

Proposition 2 Let $(N, M_0), N = (P, T, F)$, be a net system, $S = \{p_x, \ldots, p_y\}$ be a controllable siphon which can be emptied, and $\widetilde{S} = \{p_\alpha, \ldots, p_\beta\}$ be a subset of P, where $\{\alpha, \ldots, \beta\} \cap \{x, \ldots, y\} = \emptyset$. Add a monitor V_S to N and the new net system is denoted by (N', M_0') such that $I = (0, \ldots, 0, 1_{p_x}, \ldots, 1_{p_y}, 0, \ldots, 0, -b_{p_\alpha}, \ldots, -b_{p_\beta}, 0, \ldots, 0, -1_{V_S})^T$ is a P-invariant of N', where $\forall i \in \{\alpha, \ldots, \beta\}, b_{p_i} \in \mathbf{N}$; $\forall p \in P$, $M_0'(p) = M_0(p)$. Let $M_0'(V_S) = M_0(S) - \xi_S$. S is controlled if $\sum_{i\in\{\alpha,\ldots,\beta\}} b_{p_i} \cdot M_0'(p_i) < \xi_S$ holds.

Proof: If $\sum_{i\in\{\alpha,\ldots,\beta\}} b_{p_i} \cdot M_0'(p_i) < \xi_S$ holds, we have $0 > \sum_{i\in\{\alpha,\ldots,\beta\}} b_{p_i} \cdot M_0'(p_i) - \xi_S$. Thus $M_0(S) > \sum_{i\in\{\alpha,\ldots,\beta\}} b_{p_i} \cdot M_0'(p_i) + M_0(S) - \xi_S$. This implies the truth of $M_0'(S) = M_0(S) > \sum_{i\in\{\alpha,\ldots,\beta\}} b_{p_i} \cdot M_0'(p_i) + M_0'(V_S)$.

It is clear that $\{p \mid I(p) > 0\} \subseteq S$ holds. Furthermore, $\forall M \in R(M, M_0)$, $I^T \cdot M = M_0'(S) - \sum_{i\in\{\alpha,\ldots,\beta\}} b_{p_i} \cdot M_0(p_i) - M_0'(V_S)$. Obviously, we can conclude that $\forall M \in R(N, M_0), I^T \cdot M > 0$ holds. This leads to the truth of this proposition. □

For a contollable siphon S in (N, M_0), we call ξ_S the control depth variable of S. Clearly, $0 < \xi_S < M_0(S)$ holds. We can see that a siphon can be invariant-controlled by changing its control depth variable. Concerning the controllability of redundant siphons, we have the following results. In the following discussions, assume that $S_i = \{p_{x_i}, \ldots, p_{y_i}\}$, and $\{\alpha_i, \ldots, \beta_i\} \subseteq \{1, 2, \ldots, \mid P \mid\}$.

Theorem 3 Let $(N, M_0), N = (P, T, F)$, be a net system and S be a strict redundant siphon of N. Let S_1, S_2, …, and S_n be the elementary siphons of S and we have $\eta_S = a_1 \cdot \eta_{S_1} + a_2 \cdot \eta_{S_2} + \ldots + a_n \cdot \eta_{S_n}$. S is invariant-controlled if (1) $\forall i \in \{1, 2, \ldots, n\}$, $I_i = (0, \ldots, 0, 1_{p_{x_i}}, \ldots, 1_{p_{y_i}}, -b_{p_{\alpha_i}}, \ldots, -b_{p_{\beta_i}}, 0, \ldots, 0)^T$ is a P-invariant of N and (2)$M_0(S) > \sum_{u\in\{\alpha_1,\ldots,\beta_1\}} a_1 \cdot b_{p_u} \cdot M_0(p_u) + \sum_{u\in\{\alpha_2,\ldots,\beta_2\}} a_2 \cdot b_{p_u} \cdot M_0(p_u) + \ldots + \sum_{u\in\{\alpha_n,\ldots,\beta_n\}} a_n \cdot b_{p_u} \cdot M_0(p_u)$, where $\forall i \in \{1, 2, \ldots, n\}, S_i = \{p_{x_i}, \ldots, p_{y_i}\}, \{\alpha_i, \ldots, \beta_i\} \subseteq \{1, 2, \ldots, \mid P \mid\}, \forall u \in \{\alpha_i, \ldots, \beta_i\}, b_{p_u} \in \mathbf{N}$.

Proof: $\forall i \in \{1, 2, \ldots, n\}$, we have $S_i = \{p_{x_i}, \ldots, p_{y_i}\}$ and $I_i = (0, \ldots, 0, 1_{p_{x_i}}, \ldots, 1_{p_{y_i}}, 0, \ldots, 0, -b_{p_{\alpha_i}}, \ldots, -b_{p_{\beta_i}}, 0, \ldots, 0)^T$. Let $I_i = \lambda_{S_i} + \tilde{I}_i$, where $\lambda_{S_i}=(0, \ldots, 0, 1_{p_{x_i}}, \ldots, 1_{p_{y_i}}, 0, \ldots, 0)^T$ and $\tilde{I}_i=(0, \ldots, 0, -b_{p_{\alpha_i}}, \ldots, -b_{p_{\beta_i}}, 0, \ldots, 0)^T$. Let $\widehat{I}_i = a_i \cdot I_i$. Then $\forall i \in \{1, 2, \ldots, n\}, \widehat{I}_i$ is a P-invariant of N since I_i is a P-invariant of N. Let $I = \lambda_S + a_1 \cdot \tilde{I}_1 + a_2 \cdot \tilde{I}_2 + \ldots + a_n \cdot \tilde{I}_n$. We prove that I is a P-invariant of N.

$I^T \cdot [N] = \lambda_S^T \cdot [N] + a_1 \cdot \tilde{I}_1^T \cdot [N] + a_2 \cdot \tilde{I}_2^T \cdot [N] + \ldots + a_n \cdot \tilde{I}_n^T \cdot [N] = \eta_S^T + a_1 \cdot \tilde{I}_1^T \cdot [N] + a_2 \cdot \tilde{I}_2^T \cdot [N] + \ldots + a_n \cdot \tilde{I}_n^T \cdot [N] = a_1 \cdot \eta_{S_1}^T + a_2 \cdot \eta_{S_2}^T + \ldots + a_n \cdot \eta_{S_n}^T + a_1 \cdot \tilde{I}_1^T \cdot [N] + a_2 \cdot \tilde{I}_2^T \cdot [N] + \ldots + a_n \cdot \tilde{I}_n^T \cdot [N] = (a_1 \cdot \lambda_{S_1}^T \cdot [N] + a_1 \cdot \tilde{I}_1^T \cdot [N]) + (a_2 \cdot \lambda_{S_2}^T \cdot [N] + a_2 \cdot \tilde{I}_2^T \cdot [N]) + \ldots + (a_n \cdot \lambda_{S_n}^T \cdot [N] + a_n \cdot \tilde{I}_n^T \cdot [N]) = a_1 \cdot (\lambda_{S_1}^T + \tilde{I}_1^T) \cdot [N] + a_2 \cdot (\lambda_{S_2}^T + \tilde{I}_2^T) \cdot [N] + \ldots + a_n \cdot (\lambda_{S_n}^T + \tilde{I}_n^T) \cdot [N] = \sum_{i=1}^n \widehat{I}_i^T \cdot [N] = \mathbf{0}^T$. Hence I is a P-invariant of N.

$\forall M \in R(N, M_0)$, $I^T \cdot M = I^T \cdot M_0 = M_0(S) - (\sum_{u\in\{\alpha_1,\ldots,\beta_1\}} a_1 \cdot b_{p_u} \cdot M_0(p_u) + \sum_{u\in\{\alpha_2,\ldots,\beta_2\}} a_2 \cdot b_{p_u} \cdot M_0(p_u) + \ldots + \sum_{u\in\{\alpha_n,\ldots,\beta_n\}} a_n \cdot b_{p_u} \cdot M_0(p_u))$. It is clear to see that

$I^T \cdot M > 0$. Furthermore, it is true that $\{p \mid I(p) > 0\} \subseteq S$ holds. Therefore, S is an invariant-controlled siphon. □

This result shows that the controllability of a strict redundant siphons does not necessarily mean that its every elementary siphon is invariant-controlled. However, its elementary siphons must be potentially invariant-controlled.

Theorem 4 Let $(N, M_0), N = (P, T, F)$, be a net system and S be a slack redundant siphon of N. Let S_1, S_2, $\ldots$, S_n, S_{n+1}, S_{n+2}, $\ldots$, and $S_{n+m}(n \geq 2, m \geq 1)$ be the elementary siphons of S and $\eta_S = (a_1 \cdot \eta_{S_1} + a_2 \cdot \eta_{S_2} + \ldots + a_n \cdot \eta_{S_n}) - (a_{n+1} \cdot \eta_{S_{n+1}} + a_{n+2} \cdot \eta_{S_{n+2}} + \ldots + a_{n+m} \cdot \eta_{S_{n+m}})$. S is controlled if (1) $\forall i \in \{1, 2, \ldots, n\}$, $I_i = (0, \ldots, 0, 1_{p_{x_i}}, \ldots, 1_{p_{y_i}}, 0, \ldots, 0, -b_{p_{\alpha_i}}, \ldots, -b_{p_{\beta_i}}, 0, \ldots, 0)^T$ is a P-invariant of N and (2)$M_0(S) > \sum_{u \in \{\alpha_1, \ldots, \beta_1\}} a_1 \cdot b_{p_u} \cdot M_0(p_u) + \sum_{u \in \{\alpha_2, \ldots, \beta_2\}} a_2 \cdot b_{p_u} \cdot M_0(p_u) + \ldots + \sum_{u \in \{\alpha_n, \ldots, \beta_n\}} a_n \cdot b_{p_u} \cdot M_0(p_u)$, where $\forall i \in \{1, 2, \ldots, n\}$, $S_i = \{p_{x_i}, \ldots, p_{y_i}\}, \{\alpha_i, \ldots, \beta_i\} \subseteq \{1, 2, \ldots, \mid P \mid\}, \forall u \in \{\alpha_i, \ldots, \beta_i\}, b_{p_u} \in \mathbf{N}$.

Proof: It is easy to see that $\forall i \in \{1, 2, \ldots, n\}$, the maximal number of tokens removed from S due to the token flow from S_i is $\sum_{u \in \{\alpha_i, \ldots, \beta_i\}} a_i \cdot b_{p_u} \cdot M_0(p_u)$. Therefore, the maximal number of tokens removed from S can be $\sum_{u \in \{\alpha_1, \ldots, \beta_1\}} a_1 \cdot b_{p_u} \cdot M_0(p_u) + \sum_{u \in \{\alpha_2, \ldots, \beta_2\}} a_2 \cdot b_{p_u} \cdot M_0(p_u) + \ldots + \sum_{u \in \{\alpha_n, \ldots, \beta_n\}} a_n \cdot b_{p_u} \cdot M_0(p_u)$.

Since $\eta_S = (a_1 \cdot \eta_{S_1} + a_2 \cdot \eta_{S_2} + \ldots + a_n \cdot \eta_{S+n}) - (a_{n+1} \cdot \eta_{S_{n+1}} + a_{n+2} \cdot \eta_{S_{n+2}} + \ldots + a_{n+m} \cdot \eta_{S_{n+m}})$, we have $\eta_S + a_{n+1} \cdot \eta_{S_{n+1}} + a_{n+2} \cdot \eta_{S_{n+2}} + \ldots + a_{n+m} \cdot \eta_{S_{n+m}} = a_1 \cdot \eta_{S_1} + a_2 \cdot \eta_{S_2} + \ldots + a_n \cdot \eta_{S+n}$.

We consider the worst case that only tokens in S are removed when tokens in S_1, S_2, $\ldots$, and S_n are removed. Hence if $M_0(S) > \sum_{u \in \{\alpha_1, \ldots, \beta_1\}} a_1 \cdot b_{p_u} \cdot M_0(p_u) + \sum_{u \in \{\alpha_2, \ldots, \beta_2\}} a_2 \cdot b_{p_u} \cdot M_0(p_u) + \ldots + \sum_{u \in \{\alpha_n, \ldots, \beta_n\}} a_n \cdot b_{p_u} \cdot M_0(p_u)$ holds, S can never be emptied. □

Theorem 3(4) shows that under some conditions a redundant siphon can be controlled. Therefore, when elementary siphon S_i is controlled by P-invariant $I_i = (0, \ldots, 0, 1_{p_{x_i}}, \ldots, 1_{p_{y_i}}, 0, \ldots, 0, -b_{p_{\alpha_i}}, \ldots, -b_{p_{\beta_i}}, 0, \ldots, 0)^T$ and conditions (1) and (2), stated in Theorem 3(4), hold, we can conclude that redundant siphon S is implicitly controlled. As we have already known, the number of elementary siphons is much smaller than that of redundant siphons, particularly in structurally complex Petri nets. Hence, Theorem 3(4) provides an effective way to prevent a large number of siphons from being emptied by making a small number of siphons invariant-controlled. Note that Theorems 3 and 4 are sufficient conditions for the controllability of strict and slack redundant siphons, respectively. Theorem 4 is rather conservative. That means that in many cases a redundant siphon is controlled even if the conditions in Theorem 4 fail. Note that the controllability results of redundant siphons in [19] are the special cases of Theorems 3 and 4.

The presented results are quite useful. When designing a liveness-enforcing Petri net supervisor for a plant, emptiable siphons can be distinguished in the plant model by elementary and redundant ones. Then we can make the elementary siphons invariant-controlled using an active siphon control method, say by adding monitors. Therefore the redundant siphons can possibly be controlled by adjusting siphon control depth variables when necessary. In other words, if a plant Petri net model contains emptiable siphons, we need control only those elementary ones-a small fraction, but not all.

5 Identification of Elementary Siphons

According to the definition of elementary siphons, the set of elementary siphons is generally not unique in a given Petri net structure unless its $[\eta]$ is of full-rank. Our investigation reveals that, in order to prevent deadlock from occurring, how to choose the set of elementary siphons in a net system is a crucial problem. For deadlock control purpose, we propose an algorithm to determine the set of elementary siphons. From Theorem 3 or 4, we want to choose the set such that it is easier to satisfy the controllability conditions under which a redundant siphon can be always marked.

Algorithm: Identification of elementary siphons for deadlock control purpose
Input: $(N, M_0), N = (P, T, F)$.
Output: Π_E, the set of elementary siphons in (N, M_0).

Step 1. Find the set of siphons $\Pi = \{S_1, S_2, \ldots, S_k\}$ of N.
Step 2. Construct the characteristic T-vector matrix $[\eta]$ of siphons in N.
Step 3. Let $r = R([\eta])$, where $R([\eta])$ is the rank of $[\eta]$.
Step 4. Compute the initial number of tokens in $S_i, \forall i \in \{1, 2, \ldots, k\}$. We get a token sequence $M_0(S_1), M_0(S_2), \ldots, M_0(S_k)$.
Step 5. By merge sort algorithm, sort the token sequence to be $M_0(S_u)$, $M_0(S_v)$, ..., $M_0(S_q)$ in ascending order, where $\{u, v, \ldots, q\} = \{1, 2, \ldots, k\}$.
Step 6. Construct $[\eta]^R$, ordered characteristic T-vector matrix we call, where $[\eta]^R = [\eta_{S_u} \mid \eta_{S_v} \mid \ldots \mid \eta_{S_q}]^T$.
Step 7. Let $\chi_1 = \eta_{S_u}$, $\chi_2 = \eta_{S_v}$, ..., $\chi_k = \eta_{S_q}$. We have $[\eta]^R = [\chi_1 \mid \chi_2 \mid \ldots \mid \chi_k]^T$.
Step 8. Let $A = [\chi_1 \mid \chi_2 \mid \ldots \mid \chi_r]^T$.
Step 9. $\beta := r; i := 0$.
Step 10. Compute $R(A)$.
Step 11. $\alpha := r - R(A)$.
Step 12. **If** $\alpha = 0$ **then** $\Pi_E := \{S_u, \ldots, S_x\}$, where $\mid \{u, \ldots, x\} \mid = r$, $\eta_{S_u} = \chi_1, \ldots, \eta_{S_x} = \chi_\beta$; go to Step 13.
else renew matrix A via replacing the last α rows in A, $\chi^T_{\beta-\alpha+1}$, ..., $\chi^T_{\beta-1}$, and χ^T_β by $\chi^T_{\beta+1}$, ..., $\chi^T_{\beta+\alpha-1}$ and $\chi^T_{\beta+\alpha}$ in $[\eta]^R$, respectively; $\beta := \beta + \alpha$; $i := i + 1$; go to Step 10.
endif
Step 13. Output Π_E.

This algorithm will terminate and when it does, we have $i \leq k$. Next we discuss the complexity for this algorithm. It is known that the complexity of the merge sort algorithm for a sequence with k elements is $O(k \log_2 k)$. In addition, we have to compute the rank of matrix A in order to find the expected set of elementary siphons. However, as variable i indicates, the number of times of computing $R(A)$ is bounded by k, the number of minimal siphons in a net. Thus, the complexity of the algorithm is $O(k + k \log_2 k)$. Note that the complexity of this algorithm has nothing to do with the initial marking for a given net structure. This is because siphons are purely a structural object of a net system.

Note that if there are several different siphons in a net system with the identical characteristic T-vector, this algorithm selects the one as an elementary siphon, which possesses the minimal number of tokens. For example, if S_1, S_2, and S_3 are siphons with $\eta_{S_1} = \eta_{S_2} = \eta_{S_3}$ and $M_0(S_1) = 3$, $M_0(S_2) = 5$, and $M_0(S_3) = 3$, then either S_1 or S_3 is slected into the set of elementary siphons. Again, this is reasonable since from Corollary 3, we can see that S_2 is a nominal siphon which can always be marked. That S_1 and S_3 are poor siphons means if one of them is controlled, so is the other. Of course, a poor siphon does not exclude the possibility that it is a nominal one.

There may be other criteria to determine the set of elementary siphons in a net system for deadlock control purpose. This issue remains to be explored and its detailed discussion is beyond this paper.

6 Example

The Petri net system (N, M_0) shown in Figure 1 is employed to illustrate the concepts and techniques proposed in this paper. There are 11 siphons which are the supports of

P-invariants and can be always marked. And there are 15 strict minimal siphons which are S_1={p_2, p_5, p_{13}, p_{15}, p_{18}}, S_2={p_5, p_{13}, p_{14}, p_{15}, p_{18}}, S_3={p_2, p_7, p_{11}, p_{13}, p_{16}, p_{17}, p_{18}, p_{19}}, S_4={p_5, p_6, p_{11}, p_{12}, p_{21}, p_{22}}, S_5={p_5, p_{11}, p_{13}, p_{14}, p_{18}, p_{20}}, S_6={p_7, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}, p_{18}, p_{19}}, S_7={p_2, p_7, p_{13}, p_{15}, p_{16}, p_{17}, p_{18}, p_{19}}, S_8={p_7, p_{12}, p_{15}, p_{17}, p_{19}, p_{21}, p_{22}}, S_9={p_7, p_{11}, p_{12}, p_{17}, p_{19}, p_{21}, p_{22}}, S_{10}={p_7, p_{11}, p_{13}, p_{14}, p_{16}, p_{17}, p_{18}, p_{19}, p_{20}}, S_{11}={p_5, p_6, p_{12}, p_{15}, p_{21}, p_{22}}, S_{12}={p_7, p_{12}, p_{15}, p_{17}, p_{19}, p_{20}, p_{22}}, S_{13}={p_7, p_{11}, p_{12}, p_{17}, p_{19}, p_{20}, p_{22}}, S_{14}={p_5, p_6, p_{12}, p_{15}, p_{20}, p_{22}}, and S_{15}={p_5, p_6, p_{11}, p_{12}, p_{20}, p_{22}}. By integer programming or Lenstra's algorithm, it is easy to verify that S_{15}={p_5, p_6, p_{11}, p_{12}, p_{20}, p_{22}} is a nominal siphon since $min\{\sum_{p\in S_{15}} M(p) \mid M \in R(N, M_0)\} = 1$. Similarly, S_6, S_7, S_{10}, S_{11}, and S_{14} are verified to be nominal.

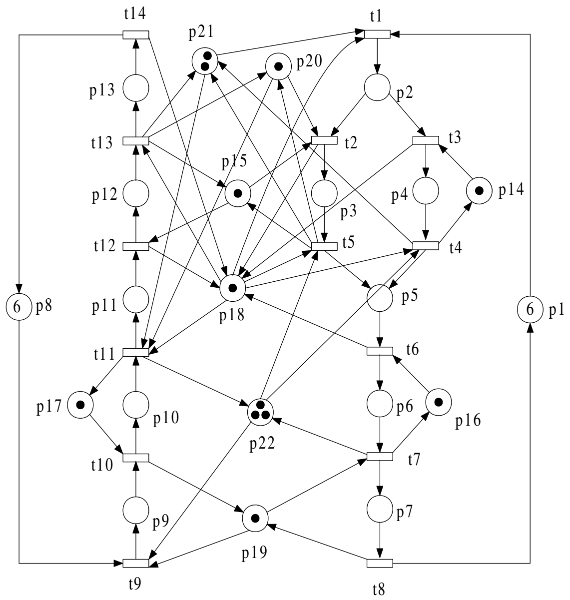

Figure 1: A Petri net system (N, M_0).

$$[\eta] = \begin{bmatrix}
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 2 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & -2 & 0 & -1 & 1 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & -2 & 0 & 2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

The characteristic T-vector matrix $[\eta]$ of the 15 strict minimal siphons of the net is shown as above. The ith row corresponds to the ith strict minimal siphon. We have $\eta_{S_2} = \eta_{S_5} = (-1, 0, 0, 1, 1, 0, 0, 0, 0, 0, -1, 0, 1, 0)^T$ and η_{S_6}=$\eta_{S_{10}}$=(−1, 0, 0, 0, 0, 0, 1, 0, −1, 0, 0, 0, 1, 0)^T. Hence we have $S_2 \cong S_5$ and $S_6 \cong S_{10}$. There are 13 sets of equivalent strict minimal siphons, which are $\langle S_1\rangle = \{S_1\}$, $\langle S_2\rangle = \{S_2, S_5\}$, $\langle S_3\rangle = \{S_3\}$, $\langle S_4\rangle = \{S_4\}$,

$\langle S_6\rangle = \{S_6, S_{10}\}$, $\langle S_7\rangle = \{S_7\}$, $\langle S_8\rangle = \{S_8\}$, $\langle S_9\rangle = \{S_9\}$, $\langle S_{11}\rangle = \{S_{11}\}$, $\langle S_{12}\rangle = \{S_{12}\}$, $\langle S_{13}\rangle = \{S_{13}\}$, $\langle S_{14}\rangle = \{S_{14}\}$, and $\langle S_{15}\rangle = \{S_{15}\}$. It is easy to see that $M_0(S_2)$=$M_0(S_5)$=3 and $M_0(S_6) = M_0(S_{10}) = 6$. Hence we say that S_2 and S_5 are poor siphons. So are S_6 and S_{10}. There is no rich siphon in $\langle S_2\rangle$ and $\langle S_6\rangle$.

By the algorithm for finding elementary siphons in a net system, we have S_1, S_2, S_3, and S_{15} as elementary siphons. It can be easily seen that η_{S_4}=η_{S_2}+$\eta_{S_{15}}$-η_{S_1}, η_{S_5}=η_{S_2}, η_{S_6}=η_{S_2}+η_{S_3}, η_{S_7}=η_{S_1}+η_{S_3}, η_{S_8}=η_{S_2}+η_{S_3}+$\eta_{S_{15}}$, η_{S_9}=η_{S_2}+η_{S_3}+$\eta_{S_{15}}$-η_{S_1}, $\eta_{S_{10}}$=η_{S_2}+η_{S_3}, $\eta_{S_{11}}$=$\eta_{S_2} + \eta_{S_{15}}$, $\eta_{S_{12}}$=$\eta_{S_1} + \eta_{S_3} + \eta_{S_{15}}$, $\eta_{S_{13}}$=$\eta_{S_3} + \eta_{S_{15}}$, and $\eta_{S_{14}}$=$\eta_{S_1} + \eta_{S_{15}}$. Hence we can say that S_5, S_6, S_7, S_8, S_{10}, S_{11}, S_{12}, S_{13}, and S_{14} are strict redundant siphons. And S_4 and S_9 are slack redundant ones.

Let $I_1 = (0,1,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,1,0,-1,0,0)^T$. We have $I_1^T \cdot [N] = \mathbf{0}^T$, $\{p \mid I_1(p) > 0\} \subseteq S_1$, and $I_1^T \cdot M_0 = M_0(p_2) + M_0(p_5) + M_0(p_{13}) + M_0(p_{15}) + M_0(p_{18}) - M_0(p_{20}) = 1 > 0$. Then S_1 is invariant-controlled. Let $I_2 = (0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ -1,\ 0)^T$ and $I_3 = (0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1,0,0,-1)^T$. Clearly, we have $I_2^T \cdot [N] = \mathbf{0}^T$, $I_3^T \cdot [N] = \mathbf{0}^T$, $\{p \mid I_2(p) > 0\} \subseteq S_2$, $\{p \mid I_3(p) > 0\} \subseteq S_3$, $I_2^T \cdot M_0$=$M_0(p_5)$+$M_0(p_{13})$+$M_0(p_{14})$+$M_0(p_{15})$+$M_0(p_{18})$-$M_0(p_{21}) = 1 > 0$, and $I_3^T \cdot M_0$=$M_0(p_2)$+$M_0(p_7)$+$M_0(p_{11})$+$M_0(p_{13})$+$\sum_{i=16}^{19} M_0(p_i)$-$M_0(p_{22}) = 4 - 3 > 0$. Therefore, both S_2 and S_3 are invariant-controlled siphons. Next we check the controllability of redundant siphons S_6 and S_7. Note that $\eta_{S_6} = \eta_{S_2} + \eta_{S_3}$, S_2 is controlled by I_2, $\{p \mid I_2(p) < 0\} = \{p_{21}\}$, S_3 is controlled by I_3, $\{p \mid I_3(p) < 0\} = \{p_{22}\}$, and $M_0(S_6) > M_0(p_{21}) + M_0(p_{22})$. By Theorem 3, S_6 is invariant-controlled, and the controllability of S_6 implies that of S_{10} due to $S_6 \cong S_{10}$. Similarly, we can verify S_7 is invariant-controlled as well. This is why S_6 and S_7 are nominal siphons.

7 Application

A hypothetical flexible manufacturing system (FMS) has seven machine tools, M1–7. Each machine tool can process two parts at the same time. Also the FMS has five robots R1–5 and each of them can hold one part. Parts enter the system through five loading buffers I1–5, and leave the system through five unloading buffers O1–5. Five part types P1–5 can be produced. Their respective production routes are as follows:

P1: I1→R4→M6→R2→M5→R2→O1 or I1→R4→M4→R3→M2→R2→O1;

P2:I2→R5→M7→R1→M1→R1→O2;

P3:I3→R1→M1→R2→M2→R3→O3 or I3→R1→M3→R3→M4→R3→O3;

P4: I4→R3→M4→R3→M3→R1→O4; and

P5: I5→R2→M6→R2→ M2→R5→O5.

The Petri net model of the FMS is shown in Figure 2, which is an S^3PR, a class of Petri net models first proposed in [7].

There are 87 strict minimal siphons which are all controllable, and 17 of them are the supports of P-invariants, which are initially marked. Thus, one can see that there are 70 strict minimal siphons which can be unmarked during the evolution of the net system. We distinguish them by elementary and redundant ones. As mentioned above, our goal is to render all strict minimal siphons controlled through making elementary siphons invariant-controlled.

By the elementary siphon identification algorithm proposed in Section 5, we find that there are 10 elementary siphons S_1, S_2, ..., and S_{10} in Figure 2. The number of strict and slack redundant siphons in the plant Petri net model are 46 and 14, respectively, (detailed in Appendix A). The 10 elementary siphons are as follows:

$S_1 = \{p_9, p_{19}, p_{28}, p_{39}, p_{45}\}$,

$S_2 = \{p_7, p_9, p_{11}, p_{14}, p_{17}, p_{21}, p_{25}, p_{30}, p_{36}, p_{37}, p_{38}, p_{39}, p_{42}, p_{43}, p_{44}\}$,

$S_3 = \{p_5, p_{11}, p_{14}, p_{25}, p_{30}, p_{38}, p_{43}\}$,

$S_4 = \{p_2, p_5, p_{11}, p_{15}, p_{17}, p_{25}, p_{30}, p_{36}, p_{37}, p_{38}, p_{41}, p_{42}, p_{43}, p_{48}\}$,

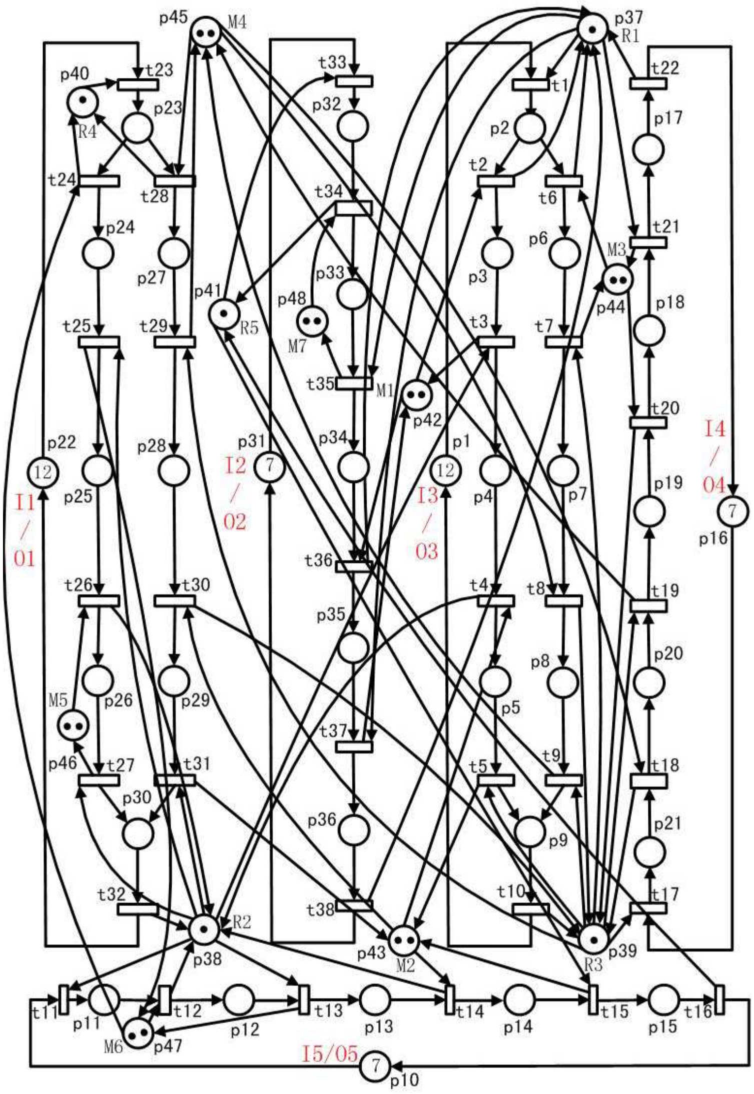

Figure 2: The Petri net system (N_0, M_0) of an FMS.

$S_5 = \{p_4, p_{13}, p_{25}, p_{30}, p_{38}, p_{47}\}$,
$S_6 = \{p_4, p_{11}, p_{13}, p_{30}, p_{38}, p_{46}\}$,
$S_7 = \{p_2, p_3, p_{17}, p_{36}, p_{37}, p_{42}\}$,
$S_8 = \{p_7, p_9, p_{18}, p_{21}, p_{28}, p_{39}, p_{44}\}$,
$S_9 = \{p_7, p_9, p_{14}, p_{19}, p_{21}, p_{29}, p_{39}, p_{43}\}$, and
$S_{10} = \{p_3, p_6, p_{17}, p_{36}, p_{37}, p_{42}, p_{44}\}$,

Ezpeleta et al. [7] developed an approach which can prevent a strict minimal siphon in an S^{3}PR from being emptied by adding a monitor for it. Here we employe this siphon control method to prevent only elementary siphons from being unmarked. As mentioned above, the controllability of redundant siphons can be ensured by making their elementary siphons invariant-controlled and modifying their control depth variables.

For these 10 elementary siphons, we add 10 monitors V_{S_1}, V_{S_2}, V_{S_3}, V_{S_4}, V_{S_5}, V_{S_6}, V_{S_7}, V_{S_8}, V_{S_9}, and $V_{S_{10}}$ to the plant model. $\forall i \in \{1, 2, \ldots, 10\}$, it is easy to see that elementary siphon S_i is invariant-controlled due to the addition of V_{S_i} when $\xi_{S_i} = 1$. By Theorems 3 and 4, it is not difficult to verify that all strict and slack redundant siphons are controlled when $\xi_{S_1} = 1$, $\xi_{S_2} = 1$, $\xi_{S_3} = 1$, $\xi_{S_4} = 1$, $\xi_{S_5} = 2$, $\xi_{S_6} = 2$, $\xi_{S_7} = 2$, $\xi_{S_8} = 2$, $\xi_{S_9} = 1$,

Table 2: Monitors added to the plant model.

monitor	$M'_0(V_{S_i})$	preset	postset
V_{S_1}	2	$t_2, t_9, t_{19}, t_{24}, t_{29}$	t_1, t_{17}, t_{23}
V_{S_2}	8	$t_5, t_7, t_{14}, t_{21}, t_{24}, t_{31}, t_{37}$	$t_1, t_{11}, t_{17}, t_{23}, t_{33}$
V_{S_3}	2	$t_4, t_6, t_{14}, t_{24}, t_{31}$	t_1, t_{11}, t_{23}
V_{S_4}	8	$t_4, t_6, t_{15}, t_{24}, t_{31}, t_{37}$	$t_1, t_{11}, t_{23}, t_{33}$
V_{S_5}	1	t_{13}, t_{25}, t_{28}	t_{11}, t_{23}
V_{S_6}	1	t_{27}, t_{28}	t_{23}
V_{S_7}	1	t_{37}	t_{33}
V_{S_8}	1	t_2, t_7, t_{20}	t_1, t_{17}
V_{S_9}	2	t_5, t_6, t_{24}, t_{30}	t_1, t_{23}
$V_{S_{10}}$	3	t_2, t_6, t_{21}, t_{37}	t_1, t_{17}, t_{33}

and $\xi_{S_{10}} = 2$. We use (N', M'_0) to denote the augmented net system after 10 monitors are added. Talbe 2 shows the number of initial tokens, preset, and postset of each monitor. The augmented net system is hence live [7] since there is no emptiable siphon anymore. By the concept of elementary siphons, only 10 additional places and 66 arcs are added in the design of monitor-based liveness enforcing Petri net supervisor for the plant model. However, using the method in [7], 70 monitors and 482 arcs have to be added.

Table 3: Effectiveness comparison for different-scale examples

Plant & Supervisor information	Example 1	Example 2	Example 3
Resources	three machines; two robot	four machines; three robots	seven machines; five robots
No. part types produced	two	three	five
Net model size	|P|=15, |T|=11, |F|=39	|P|=26, |T|=20, |F|=74	|P|=48, |T|=38, |F|=142
No. monitors & arcs added using the method [7]	3 monitors 15 arcs	18 monitors 106 arcs	70 monitors 482 arcs
No. monitors & arcs added using elementary siphons	2 monitors 10 arcs	6 monitors 32 arcs	10 monitors 66 arcs

To show the importance of applying the concept of elementary siphons for a particular deadlock prevention policy, three different-scale FMS examples whose net models are all S^3PR are used to illustrate the effectiveness in reducing the structural complexity of monitor-based liveness-enforcing Petri net supervisors. We choose S^3PR so that the deadlock prevention policy proposed in [7] can be applied. The first and the second examples are the net systems shown in Figure 3(a) and Figure 8 in [7], respectively. The third example is the one we used in this section. Table 3 shows that large systems can phenomenalize the significance of the concept of elementary siphons in effective deadlock control.

8 Concluding Remarks

Investigations on deadlock resolution in computer-integrated systems and networks have received much attention for a long time [9]. Deadlocks, a highly undesired phenomenon, often lead to catastrophic results in these systems. Their efficient handling becomes a necessary condition for a system to gain high safeness, reliability, or productivity. This paper presents an elementary siphon theory and opens a new way to design a liveness-enforcing Petri net supervisor for discrete event systems.

Since Petri nets are well suited for describing the sequence of events that trace the dynamics of a discrete event system, they have become the most extensively used tools to deal with deadlock problems in these systems. A major breakthrough to systematically evaluate the liveness of the Petri net models for various resource allocation systems, and to synthesize effective and computationally efficient liveness enforcing Petri net supervisors is the formal characterization of the non-liveness of Petri nets through the formation of a particular Petri net structural object, known as siphons [23] [7].

Based on siphons, deadlock prevention can be achieved by adding monitors to the plant Petri net model to constraint the behavior of the modeled system such that deadlocks can never occur. The significance of this strategy lies in that both the plant model of a system and its supervisor are unified in the same framework: Petri nets. In some deadlock avoidance techniques, the system model is a Petri net while its supervisor is an automaton. However, a serious drawback of the existing approaches is that the resulting Petri net supervisor can easily reach an unmanageable level. It is so because the number of minimal siphons in a Petri net is theoretically exponential with its size, and for each problematic minimal siphon, a monitor has to be added to the plant model to prevent the siphon from being unmarked. The discovery and investigation of elementary siphons in Petri nets are motivated by the urgent need for effective ways to deal with the structural complexity problem of monitor-based liveness enforcing Petri net supervisors.

At present, the number of elementary siphons is bounded by both place and transition count in a net. A redundant siphon can be possibly controlled if its elementary siphons are invariant-controlled. A number of issues require further research. One is the net analysis according to elementary siphons. Specifically, we can, based on elementary siphons, re-expound the existing results which have been expounded by means of siphons. For example, the dynamic behavior of free choice nets and asymmetric choice nets are extensively investigated in terms of the characterization of siphons [6] [5]. Thus a naturally arising problem is the reformulation of the behavioral properties of these nets in terms of elementary siphons.

In addition, it is of significance to investigate the elementary siphon related problems in some Petri net subclasses, such as augmented marked graph [5], S^3PR [7], RCN-merged nets [13], ERCN-merged nets [26], PNR [14], ERCN* merged nets [16], etc. These problems may conclude the relationship between the rank of a net's incidence matrix and the number of its elementary siphons, and the relationship between a redundant siphon and its elementary ones. For example, our preliminary investigation [20] has shown that in an S^3PR, a redundant siphon can be composed by its elementary ones, and marked by modifying the control depth variables of its elementary siphons provided that they are invariant-controlled. That means that for a marked S^3PR, a liveness enforcing Petri net supervisor can be always obtained by just controlling its elementary siphons. We have developed a polynomial algorithm to find a set of elementary siphons in an S^3PR without knowing all siphons beforehand [20]. This allows us to obtain every redundant siphon by some simple arithmetic operations on a few elementary siphons.

The theory of elementary siphons looks promising to tackle the complexity problem in the design of monitor-based liveness enforcing Petri net supervisors, particularly for the large plant Petri net models. Future research is to develop some novel deadlock prevention policies which can guarantee that a small number of monitors are added and most permissive behavior can be allowed. For example, the iterative control policy for deadlock prevention proposed in [18] [12] can be reformulated by only considering elementary siphons in a net system. At each iteration step, we add a monitor for each problematic elementary siphon. Hopefully, a structurally simple Petri net supervisor is obtained while achieving the same deadlock prevention purpose.

For a particular net system, how to optimally select the set of elementary siphons remains to be an important but open issue such that under a given deadlock control policy, the behavior of the controlled system is as permissive as possible. As seen in this paper, we propose a criterion to select elementary siphons that the number of tokens in an elementary one is as small as possible. Another criterion is to make the number of slack redundant ones as small as possible in a net. The two criteria may supposedly coincide for some net subclasses. However, we know nothing about these subclasses as yet. Moreover, when the two criteria result in different sets of elementary siphons for a net system, new issues are produced: which one is better than the other under a particular deadlock control policy, for instance, the one proposed in [7] for S^3PR.

So far, we have concentrated our efforts on the bounded ordinary Petri nets where the weight of an arc is unit. The case of elementary siphons in a generalized Petri net is an interesting but open problem. For the controllability of siphons in a generalized Petri net, Barkaoui et al. [4] proposed the concepts of *max cs-property* and *min cs-property*. It is shown that if a marked (generalized) Petri net satisfies the *max cs-property*, then it is deadlock-free. For some special subclasses such as S^4R [1], G-task [3], and S^4PR [25], their deadlock-freedom and liveness coincide. Therefore, a noteworthy research direction is to extend the theory of elementary siphons from ordinary Petri nets to generalized ones. For a redundant siphon S in generalized net systems, a fascinating condition is strongly expected under which S is max(min)-controlled if its elementary ones are max(min)-controlled.

In the authors' opinion, the discovery of elementary siphons opens a novel avenue to the design of Petri net supervisors for discrete event systems and to the analysis of structural and dynamic behavior of Petri nets.

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A Redundant Siphons in the plant net model of Figure 2

For economy of space, we omit symbol "p" in the elements of each siphon.

$S_{11} = \{9, 15, 17, 30, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 48\}$
$S_{12} = \{9, 15, 17, 25, 30, 36, 37, 38, 39, 41, 42, 43, 44, 45, 47, 48\}$
$S_{13} = \{9, 11, 15, 17, 30, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 48\}$
$S_{14} = \{9, 11, 15, 17, 25, 30, 36, 37, 38, 39, 41, 42, 43, 44, 45, 418\}$
$S_{15} = \{9, 14, 17, 30, 36, 37, 38, 39, 42, 43, 44, 45, 46, 47\}$
$S_{16} = \{9, 14, 17, 25, 30, 36, 37, 38, 39, 42, 43, 44, 45, 47\}$
$S_{17} = \{9, 11, 14, 17, 30, 36, 37, 38, 39, 42, 43, 44, 45, 46\}$
$S_{18} = \{9, 11, 14, 17, 25, 30, 36, 37, 38, 39, 42, 43, 44, 45\}$
$S_{19} = \{9, 14, 18, 30, 38, 39, 43, 44, 45, 46, 47\}$
$S_{20} = \{9, 14, 18, 25, 30, 38, 39, 43, 44, 45, 47\}$
$S_{21} = \{9, 11, 14, 18, 30, 38, 39, 43, 44, 45, 46\}$
$S_{22} = \{9, 11, 14, 18, 25, 30, 38, 39, 43, 44, 45\}$
$S_{23} = \{9, 14, 18, 29, 39, 43, 44, 45\}$
$S_{24} = \{9, 18, 28, 39, 44, 45\}$
$S_{25} = \{9, 14, 19, 30, 38, 39, 43, 45, 46, 47\}$
$S_{26} = \{9, 14, 19, 25, 30, 38, 39, 43, 45, 47\}$
$S_{27} = \{9, 11, 14, 19, 30, 38, 39, 43, 45, 46\}$
$S_{28} = \{9, 11, 14, 19, 25, 30, 38, 39, 43, 45\}$
$S_{29} = \{9, 14, 19, 29, 39, 43, 45\}$
$S_{30} = \{7, 9, 15, 17, 21, 30, 36, 37, 38, 39, 41, 42, 43, 44, 46, 47, 48\}$
$S_{31} = \{7, 9, 15, 17, 21, 25, 30, 36, 37, 38, 39, 41, 42, 43, 44, 47, 48\}$
$S_{32} = \{7, 9, 11, 15, 17, 21, 30, 36, 37, 38, 39, 41, 42, 43, 44, 46, 48\}$
$S_{33} = \{7, 9, 11, 15, 17, 21, 25, 30, 36, 37, 38, 39, 41, 42, 43, 44, 48\}$
$S_{34} = \{7, 9, 14, 17, 21, 30, 36, 37, 38, 39, 42, 43, 44, 46, 47\}$
$S_{35} = \{7, 9, 14, 17, 21, 25, 30, 36, 37, 38, 39, 42, 43, 44, 47\}$
$S_{36} = \{7, 9, 11, 14, 17, 21, 30, 36, 37, 38, 39, 42, 43, 44, 46\}$
$S_{37} = \{5, 6, 15, 17, 30, 36, 37, 38, 41, 42, 43, 44, 46, 47, 48\}$
$S_{38} = \{5, 6, 15, 17, 25, 30, 36, 37, 38, 41, 42, 43, 44, 47, 48\}$
$S_{39} = \{5, 6, 11, 15, 17, 30, 36, 37, 38, 41, 42, 43, 44, 46, 48\}$
$S_{40} = \{5, 6, 11, 15, 17, 25, 30, 36, 37, 38, 41, 42, 43, 44, 48\}$
$S_{41} = \{7, 9, 14, 18, 21, 30, 38, 39, 43, 44, 46, 47\}$
$S_{42} = \{7, 9, 14, 19, 21, 30, 38, 39, 43, 46, 47\}$
$S_{43} = \{5, 14, 30, 38, 43, 46, 47\}$
$S_{44} = \{7, 9, 14, 18, 21, 25, 30, 38, 39, 43, 44, 47\}$
$S_{45} = \{7, 9, 14, 19, 21, 25, 30, 38, 39, 43, 47\}$
$S_{46} = \{5, 14, 25, 30, 38, 43, 47\}$
$S_{47} = \{7, 9, 11, 14, 18, 21, 30, 38, 39, 43, 44, 46\}$

$S_{48} = \{7, 9, 11, 14, 19, 21, 30, 38, 39, 43, 46\}$
$S_{49} = \{5, 11, 14, 30, 38, 43, 46\}$
$S_{50} = \{7, 9, 11, 14, 18, 21, 25, 30, 38, 39, 43, 44\}$
$S_{51} = \{7, 9, 11, 14, 19, 21, 25, 30, 38, 39, 43\}$
$S_{52} = \{3, 9, 15, 17, 29, 36, 37, 39, 41, 42, 43, 44, 45, 48\}$
$S_{53} = \{3, 9, 14, 17, 29, 36, 37, 39, 42, 43, 44, 45\}$
$S_{54} = \{3, 9, 17, 28, 36, 37, 39, 42, 44, 45\}$
$S_{55} = \{2, 9, 15, 17, 19, 30, 36, 37, 38, 39, 41, 42, 43, 45, 46, 47, 48\}$
$S_{56} = \{2, 9, 15, 17, 19, 25, 30, 36, 37, 38, 39, 41, 42, 43, 45, 47, 48\}$
$S_{57} = \{2, 7, 9, 15, 17, 19, 21, 30, 36, 37, 38, 39, 41, 42, 43, 46, 47, 48\}$
$S_{58} = \{2, 5, 15, 17, 30, 36, 37, 38, 41, 42, 43, 46, 47, 48\}$
$S_{59} = \{2, 7, 9, 15, 17, 19, 21, 25, 30, 36, 37, 38, 39, 41, 42, 43, 47, 48\}$
$S_{60} = \{2, 5, 15, 17, 25, 30, 36, 37, 38, 41, 42, 43, 47, 48\}$
$S_{61} = \{2, 9, 11, 15, 17, 19, 30, 36, 37, 38, 39, 41, 42, 43, 45, 46, 48\}$
$S_{62} = \{2, 9, 11, 15, 17, 19, 25, 30, 36, 37, 38, 39, 41, 42, 43, 45, 48\}$
$S_{63} = \{2, 7, 9, 11, 15, 17, 19, 21, 30, 36, 37, 38, 39, 41, 42, 43, 46, 48\}$
$S_{64} = \{2, 5, 11, 15, 17, 30, 36, 37, 38, 41, 42, 43, 46, 48\}$
$S_{65} = \{2, 7, 9, 11, 15, 17, 19, 21, 25, 30, 36, 37, 38, 39, 41, 42, 43, 48\}$
$S_{66} = \{4, 13, 30, 38, 46, 47\}$
$S_{67} = \{3, 7, 9, 15, 17, 21, 29, 36, 37, 39, 41, 42, 43, 44, 48\}$
$S_{68} = \{3, 7, 9, 14, 17, 21, 29, 36, 37, 39, 42, 43, 44\}$
$S_{69} = \{3, 7, 9, 17, 21, 28, 36, 37, 39, 42, 44\}$
$S_{70} = \{7, 9, 14, 18, 21, 29, 39, 43, 44\}$

The relationships of the characteristic T-vectors between redundant and elementary siphons are shown as follows.

$\eta_{S_{11}} = \eta_{S_1} + \eta_{S_4} + \eta_{S_5} + \eta_{S_6} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{12}} = \eta_{S_1} + \eta_{S_4} + \eta_{S_5} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{13}} = \eta_{S_1} + \eta_{S_4} + \eta_{S_6} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{14}} = \eta_{S_1} + \eta_{S_4} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{15}} = \eta_{S_1} + \eta_{S_2} + \eta_{S_5} + \eta_{S_6}$
$\eta_{S_{16}} = \eta_{S_1} + \eta_{S_2} + \eta_{S_5}$
$\eta_{S_{17}} = \eta_{S_1} + \eta_{S_2} + \eta_{S_6}$
$\eta_{S_{18}} = \eta_{S_1} + \eta_{S_2}$
$\eta_{S_{19}} = \eta_{S_1} + \eta_{S_3} + \eta_{S_5} + \eta_{S_6} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{20}} = \eta_{S_1} + \eta_{S_3} + \eta_{S_5} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{21}} = \eta_{S_1} + \eta_{S_3} + \eta_{S_6} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{22}} = \eta_{S_1} + \eta_{S_3} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{23}} = \eta_{S_1} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{24}} = \eta_{S_1} + \eta_{S_8}$
$\eta_{S_{25}} = \eta_{S_1} + \eta_{S_3} + \eta_{S_5} + \eta_{S_6} + \eta_{S_9}$
$\eta_{S_{26}} = \eta_{S_1} + \eta_{S_3} + \eta_{S_5} + \eta_{S_9}$
$\eta_{S_{27}} = \eta_{S_1} + \eta_{S_3} + \eta_{S_6} + \eta_{S_9}$
$\eta_{S_{28}} = \eta_{S_1} + \eta_{S_3} + \eta_{S_9}$
$\eta_{S_{29}} = \eta_{S_1} + \eta_{S_9}$
$\eta_{S_{30}} = \eta_{S_4} + \eta_{S_5} + \eta_{S_6} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{31}} = \eta_{S_4} + \eta_{S_5} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{32}} = \eta_{S_4} + \eta_{S_6} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{33}} = \eta_{S_4} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{34}} = \eta_{S_2} + \eta_{S_5} + \eta_{S_6}$
$\eta_{S_{35}} = \eta_{S_2} + \eta_{S_5}$
$\eta_{S_{36}} = \eta_{S_2} + \eta_{S_6}$
$\eta_{S_{37}} = \eta_{S_4} + \eta_{S_5} + \eta_{S_6} + \eta_{S_{10}} - \eta_{S_7}$

$\eta_{S_{38}} = \eta_{S_4} + \eta_{S_5} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{39}} = \eta_{S_4} + \eta_{S_6} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{40}} = \eta_{S_4} + \eta_{S_{10}} - \eta_{S_7}$
$\eta_{S_{41}} = \eta_{S_3} + \eta_{S_5} + \eta_{S_6} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{42}} = \eta_{S_3} + \eta_{S_5} + \eta_{S_6} + \eta_{S_9}$
$\eta_{S_{43}} = \eta_{S_3} + \eta_{S_5} + \eta_{S_6}$
$\eta_{S_{44}} = \eta_{S_3} + \eta_{S_5} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{45}} = \eta_{S_3} + \eta_{S_5} + \eta_{S_9}$
$\eta_{S_{46}} = \eta_{S_3} + \eta_{S_5}$
$\eta_{S_{47}} = \eta_{S_3} + \eta_{S_6} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{48}} = \eta_{S_3} + \eta_{S_6} + \eta_{S_9}$
$\eta_{S_{49}} = \eta_{S_3} + \eta_{S_6}$
$\eta_{S_{50}} = \eta_{S_3} + \eta_{S_8} + \eta_{S_9}$
$\eta_{S_{51}} = \eta_{S_3} + \eta_{S_9}$
$\eta_{S_{52}} = \eta_{S_1} + \eta_{S_4} + 2\eta_{S_8} + 2\eta_{S_9} + 2\eta_{S_{10}} - \eta_{S_2} - \eta_{S_7}$
$\eta_{S_{53}} = \eta_{S_1} + \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}}$
$\eta_{S_{54}} = \eta_{S_1} + \eta_{S_8} + \eta_{S_{10}}$
$\eta_{S_{55}} = \eta_{S_1} + \eta_{S_4} + \eta_{S_5} + \eta_{S_6} + \eta_{S_9}$
$\eta_{S_{56}} = \eta_{S_1} + \eta_{S_4} + \eta_{S_5} + \eta_{S_9}$
$\eta_{S_{57}} = \eta_{S_4} + \eta_{S_5} + \eta_{S_6} + \eta_{S_9}$
$\eta_{S_{58}} = \eta_{S_4} + \eta_{S_5} + \eta_{S_6}$
$\eta_{S_{59}} = \eta_{S_4} + \eta_{S_5} + \eta_{S_9}$
$\eta_{S_{60}} = \eta_{S_4} + \eta_{S_5}$
$\eta_{S_{61}} = \eta_{S_1} + \eta_{S_4} + \eta_{S_6} + \eta_{S_9}$
$\eta_{S_{62}} = \eta_{S_1} + \eta_{S_4} + \eta_{S_9}$
$\eta_{S_{63}} = \eta_{S_4} + \eta_{S_6} + \eta_{S_9}$
$\eta_{S_{64}} = \eta_{S_4} + \eta_{S_6}$
$\eta_{S_{65}} = \eta_{S_4} + \eta_{S_9}$
$\eta_{S_{66}} = \eta_{S_5} + \eta_{S_6}$
$\eta_{S_{67}} = \eta_{S_4} + 2\eta_{S_8} + 2\eta_{S_9} + 2\eta_{S_{10}} - \eta_{S_2} - \eta_{S_7}$
$\eta_{S_{68}} = \eta_{S_8} + \eta_{S_9} + \eta_{S_{10}}$
$\eta_{S_{69}} = \eta_{S_8} + \eta_{S_{10}}$
$\eta_{S_{70}} = \eta_{S_8} + \eta_{S_9}$